

# **Elements of Plasticity**

Theory and Computation

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# **Elements of Plasticity**

## Theory and Computation

**SECOND EDITION**

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## Preface to the second edition

The good reception to the first edition in academia as well as among practising engineers and the positive echo throughout the professional community have been understood as an obligation to continue caring about the contents of the book. Using *Elements of Plasticity* in the class as a companion textbook to the related course taught by the author at Stuttgart University revealed, over the years, space for improvements, now implemented in the second edition. It is anticipated that the readership will benefit even more from the revised text, which has been supplemented where advisable.

Apart from the impulses that came while teaching the subject, helpful suggestions have been contributed by readers of the book. As in the first edition, the author has pleasure in acknowledging the indispensable assistance of Grethe Knapp Christiansen in processing the text. Thanks are due to the Publishers for encouraging the project and for its efficient realization.

Ioannis Doltsinis  
Stuttgart, Germany  
January 2010

## Preface to the first edition

Plasticity, the ability to undergo permanent deformations, is a property of metallic materials that has great significance for the load carrying behaviour of engineering structures, and for the manufacturing of structural components by forming processes. This book deals with the load carrying aspect of plasticity. In particular, deformations are considered infinitesimal and emphasis is placed on the distribution and intensity of stress and permanent strain. Unlike elasticity, under plastic conditions solutions are not scalable. Instead they evolve during the course of loading, as a result of the interaction between the structural system and the changing material characteristics. The storage of stress that may accompany plastic strain determines the behaviour upon unloading and subsequent loading cycles. At the same time, the stress limiting properties of plastic materials allow for the estimation of the load carrying capability of structures, and of failure limits under time varying conditions. This situation is of vital importance to the structural analyst and to the design engineer, who are forced to establish the strength of the structure for safety under the constraints of minimum material and weight, and thus have to account for plasticity.

For the purpose of an overview to this volume, two phases will be distinguished in the evolution of the research topic of plasticity. In the first phase, researchers explored plastic and elastoplastic material behaviour and set up mathematical formalisms for its description, investigated the behaviour of elastoplastic structures, developed analytical solutions and postulated general theorems, thus establishing the theoretical foundations of plasticity. However, one fact ought not to be overlooked when considering this phase of research: the difficulty in obtaining solutions, because of the complexity of the rigorous material description, which lead to various compromising but nonetheless interesting alternative approaches. A representative stage of that era is documented in the literature (*Proceedings of the Second Symposium on Naval Structural Mechanics*, Brown University, Rhode Island, April 5–7, 1960, E.H. Lee and P.S. Symonds (Eds), Pergamon Press, Oxford, 1960).

The second phase of research, in which the author was involved, is characterized by the development of computational methods of elastoplastic analysis. The use of computational methods in structural analysis began in the 1950s and evolved into the finite element method, the boundary element method, and the other numerical methods that are widely used today. These

are not only utilized for standard investigations of stress and deformation, but also as research tools in various branches of science and engineering. Initial steps in computer-based elastoplastic analysis were explorative in nature. They combined physical and numerical constituents mostly dependent on intuition. The author, then at the Institute for Statics and Dynamics of Aerospace Structures, University of Stuttgart, has actively experienced the initiating impulses, and from the late 1960s had the opportunity to shape the development of rigorous numerical techniques of elastoplastic analysis. The prime objective was to produce stable and accurate computation schemes for application in engineering practice. Their implementation in the general-purpose, finite element software ASKA (Automatic System for Kinematic Analysis) with Dipl.-Ing. (ETH) Hans Balmer from the early 1970s was exclusively guided by industrial demands. The algorithms, continuously extended with regard to the structural elements, on one hand, and the inelastic material options (thermoplasticity, creep, viscoplasticity, soil materials), on the other hand, have also become standard procedures in the other commercial software PERMAS. The latter was developed from ASKA and has since progressed independently.

The community of practising engineers has shown, from the beginning, an interest in the advantages offered by the computational techniques of elastoplastic analysis, but at the same time scientists also showed a hesitancy in adopting the novel approach, despite its firm grounding in the basics of plasticity. Numerous seminars and workshops held at the international level served audiences from both industry and academia, and have given rise to considerable published material on the subject. On the other hand, working with graduating students has revealed a tendency among them to simply apply computer software with little regard to the circumstance. This motivated the author to establish a university course, which stressed the most important elements of plasticity to improve the physical understanding, to correctly posing problems of elastoplastic analysis, solving problems with the aid of the computer and providing a sound interpretation of the numerical results. This course is being taught by the author at the Faculty of Aerospace Engineering, University of Stuttgart, for almost two decades now. Working with students over a long period has revealed a need for *written* background material. Although some excellent monographs on plasticity are available and are referred to in this volume, the conventional treatment of the subject does not appear well suited for the purpose of a computer-oriented approach.

As a result, the present volume has been produced by bridging conventional theory and the numerical analysis of elastoplastic systems. The text focuses on the most important elements of theory and computation using matrix notation. It avoids the development of analytical solutions except for the purpose of illustration and verification. The scope of the book goes much further than the time limitations of the original, one-semester university course. In particular, it answers a more general demand for the subject,

and includes results of research and development work by the author and his team in computational plasticity, compiled from unpublished notes and from papers in professional journals. Complementary to plasticity, some considerations on creep and viscoplasticity have been added in the book, and a number of selected applications from engineering practice demonstrate the usage of computational techniques. The book is aimed equally at graduate students, practising engineers and consultants, and can serve either to elucidate computational concepts and tools for the analysis of elastoplastic structures and solids, or to further advance the essential knowledge of the subject.

The author would like to express his appreciation to WIT Press for their cooperation in publishing the book and for the care taken with its production.

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