Behavior of spatial nonlinear elastic waves in unbound domains

M. I. Bykova, N. D. Verveyko
Voronezh State University, Russia.

Abstract

The subject of the research is the propagation of isolated nonlinear elastic waves of arbitrary geometry in space. The equations in partial derivatives are obtained to describe the intensity of waves in the neighborhood of their fronts. These equations are presented as three-dimensional generalization of Korteweg-de Vries equations. The resulting solutions consider the dynamic structure of the waves.

1 Introduction

The propagation of separate nonlinear elastic waves of arbitrary geometry in space is investigated in this paper. Our goal is to construct equations in partial derivatives for transport of intensity of waves. These equations should describe the behavior of the wave in the neighborhood closest to its front.

We consider the propagation of spatial nonlinear elastic waves in unbounded space. In figure 1 the element of surface $\Sigma$ (with normal $\vec{n}$ and tangent directions $\vec{t}_1$, $\vec{t}_2$) is presented. This element propagates with velocity $G$ along normal $\vec{n}$. The assumption is made that strains and velocities change more along $\vec{n}$ then along $\vec{t}$.

Figure 2 represents the diagram of changing the velocity in the neighborhood of surface $\Sigma$. In the case of linear elastic waves, it is possible to implement the harmonic analysis or the analysis based upon the theory of propagation for discontinuity [1], which gives solution in piece-constant form in transition area. This solution is represented in figure 2 by dotted line.

To investigate the structure of elastic waves and to evaluate the effect of the
microstructure of materials, we will consider the closed system of equations in partial derivatives that allows the description of the dynamic deformation of a nonlinear elastic medium.

The assumption is done regarding the structure of the continuous medium: its microstructure is considered as a combination of material particles with equal masses; the particles are located at equal distance \( h \) from each to other and are connected by springs of equal rigidity \( c \) (Fig.3).

### 2 Nonlinear equations of motion for the elastic medium with microstructure

Let us assume that the elastic potential of the medium can be described as the polynomial of third order regarding the deformations

\[
W = \frac{1}{2} \lambda e_{kk}^2 + \frac{1}{3} \alpha \lambda e_{kk}^3 + \mu e_{ij} e_{ij} + \beta \mu e_{kk} e_{jj} e_{kl} + \frac{2}{3} \nu \mu e_{ij} e_{jk} e_{ki}.
\]  
(1)

Here \( \lambda, \mu \) - elastic parameters Lame, \( \alpha, \beta, \nu \) - parameters of nonlinear elastic material, \( e_{ij} = 1/2(u_{ij} + u_{ji}) \) - tensor of Koch’s deformations, \( u_{i} \) - components of displacement- vector.

\[
I_1 = e_{kk}; \quad I_2 = e_{ij} e_{ij}; \quad I_3 = e_{ij} e_{jk} e_{ki}.
\]

\[
I_1 = e_{kk}; \quad I_2 = e_{ij} e_{ij}; \quad I_3 = e_{ij} e_{jk} e_{ki} - \text{invariant tensors of deformation.}
\]

Tensor of deformation can be represented in a form of differences:

\[
\tilde{e}_{ij} = \frac{1}{2} \left[ \frac{u_i(x_j + h_j) - u_i(x_j - h_j)}{2h_j} + \frac{u_j(x_i + h_i) - u_j(x_i - h_i)}{2h_i} \right].
\]  
(2)

Using Tailor’s row for displacement \( u_i \) in neighborhood of point \( x \), we obtain

\[
\tilde{e}_{ij} = \left[ \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left( \frac{h^2}{6} \frac{\partial^3 u_i}{\partial x_j^3} + \frac{h^2}{6} \frac{\partial^3 u_j}{\partial x_i^3} \right) \right] + \ldots = e_{ij} + \frac{h^2}{6} e'_{ij}.
\]  
(3)

Elastic potential of considered medium has form [3].

\[
\tilde{w} = \frac{1}{2} \lambda \tilde{e}_{kk}^2 + \frac{1}{3} \alpha \lambda \tilde{e}_{kk}^3 + \mu \tilde{e}_{ij} \tilde{e}_{ij} + \beta \mu \tilde{e}_{kk} \tilde{e}_{jj} \tilde{e}_{kl} + \frac{2}{3} \nu \mu \tilde{e}_{ij} \tilde{e}_{jk} \tilde{e}_{ki}.
\]  
(4)
Strains of nonlinear elastic medium with microstructure has the following form
\[
\sigma_{ij} = \frac{\partial \ddot{w}}{\partial \ddot{\varepsilon}_{ij}} = \lambda (\dot{I}_1 + \alpha \dot{I}_1^2 + \frac{\beta \mu}{\lambda} \dot{I}_2) \delta_{ij} + 2 \mu (\dot{\varepsilon}_{ij} + \beta \ddot{I}_1 \dot{\varepsilon}_{ij} + \nu \dot{\varepsilon}_{ik} \ddot{\varepsilon}_{kj}). \tag{5}
\]
After the substitutions from (3), the equation (5) can be written as
\[
\sigma_{ij} = \lambda \{I_1 + \alpha I_1^2 + \frac{\beta \mu}{\lambda} I_2\} \delta_{ij} + \frac{\hbar^2}{12} \{I_1^* + \alpha 2I_1 I_1^* + \\
+ \beta \frac{\mu}{\lambda} 2e_y e_y^* \} \delta_{ij} + 2 \mu \{e_y + \beta I_1 e_y + \nu \dot{\varepsilon}_{ik} e_{kj}\} + \\
+ 2 \mu \frac{\hbar^2}{12} \{\beta (I_1 e_y^* + I_1^* e_y) + \nu (e_y^* e_y + e_{ik} e_{kj})\} \tag{6}
\]
The equations of movement in displacements for a nonlinear elastic medium with microstructure \(\rho \ddot{u}_i / \partial t^2 = \ddot{\sigma}_{ij,j} + \dddot{b}_i\) (with recording amounts of order \(\hbar^2\)) can be represented as
\[
\rho \frac{\partial^2 u_i}{\partial t^2} = \lambda \frac{\partial^3 u_i}{\partial x_i \partial x_j \partial x_j} + \mu \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_k} + \lambda (2 \alpha \frac{\partial u_i}{\partial x_i} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \\
+ \frac{1}{2} \beta \frac{\mu}{\lambda} \frac{\partial u_p}{\partial x_q} \frac{\partial^3 u_p}{\partial x_q \partial x_q \partial x_q} + \frac{1}{2} \beta \frac{\mu}{\lambda} \frac{\partial u_q}{\partial x_p} \frac{\partial^3 u_q}{\partial x_p \partial x_p \partial x_p} + \\
+ \frac{1}{2} \beta \frac{\mu}{\lambda} \frac{\partial u_q}{\partial x_p} \frac{\partial^3 u_p}{\partial x_q \partial x_q \partial x_q} + \frac{1}{2} \beta \frac{\mu}{\lambda} \frac{\partial u_p}{\partial x_q} \frac{\partial^3 u_q}{\partial x_p \partial x_p \partial x_p} + \\
+ 2 \mu \left(\frac{1}{2} \beta \frac{\partial u_i}{\partial x_i} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{4} \frac{\partial u_i}{\partial x_j} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{4} \frac{\partial u_j}{\partial x_i} \frac{\partial^2 u_j}{\partial x_i \partial x_i} + \\
+ \frac{1}{4} \frac{\partial u_j}{\partial x_i} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{4} \frac{\partial u_i}{\partial x_j} \frac{\partial^2 u_j}{\partial x_i \partial x_i} + \frac{1}{4} \frac{\partial u_j}{\partial x_i} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{4} \frac{\partial u_i}{\partial x_j} \frac{\partial^2 u_j}{\partial x_i \partial x_i}\right) + \\
+ \frac{\hbar^2}{12} \frac{\partial^4 u_i}{\partial x_i^4} + \mu \frac{\hbar^2}{3} \frac{\partial^4 u_i}{\partial x_i^4} + O(h^2, h^4), \tag{7}
\]
\(i, j, p, q = 1, 2, 3\).

3 Propagation of elastic waves on the condition of linear approximation

In accordance with the methods of theory of small parameters, the solution for such problems can be represented in the form of the power-mode row regarding the small parameters \(\alpha, \beta, \nu\) (\(\alpha_1 = \alpha, \ \alpha_2 = \beta, \ \alpha_3 = \nu\));
We will consider two kinds of decomposition: external and internal. External decomposition of our general problem (when \( \alpha, \beta, \gamma \to 0 \)) results in solving the dynamic problems of the linear elastic theory: the spatial shock waves of the longitudinal and shear types are propagated in the nonlinear elastic medium microstructure.

The system of successive dynamic equations for external decomposition may be formed in analogy to the equations in the Lame’s form with the right part in the elastic theory.

For the case of zero external decomposition the intensity of the spatial shock waves of jump are modified with the accordance to [2] by the low geometrical optics.

For the case of longitudinal waves, their intensity \( W^0 \) should satisfy the equation of the transfer:

\[
\frac{dW^0}{ds} - \Omega W^0 = 0, \quad \text{where } s = s_0 + G_i t, \quad W^0 = [v_i]n_i.
\]

In the case of transverse waves the equation of the transfer (9) will be:

\[
\frac{d\omega_i^0}{dl} - \Omega \omega_i^0 = 0, \quad \text{where } l = l_0 + G_s t, \quad \omega_i^0 = [v_i].
\]

\[
\Omega = \frac{\Omega_0 - K_0 s}{1 - 2\Omega_0 s + K_0 s^2}, \quad \Omega_0 = (\Omega_{1n} + \Omega_{2n})/2, \quad K_0 = \sqrt{\Omega_{1n} \Omega_{2n}}.
\]

\( \Omega_1 = 1/\rho_1, \quad \Omega_2 = 1/\rho_2 \) - general curvatures of surface, \( \rho_1, \rho_2 \) - the general radii of curvature.

The solution of the equations (9, 10) has a form

\[
W^0 / W_0^0 = (1 - 2\Omega_0 s + K_0 s^2)^{-1/2},
\]

\[
\omega_i^0 / \omega_{i0}^0 = (1 - 2\Omega_0 l + K_0 l^2)^{-1/2}.
\]

where \( W_0^0 \) and \( \omega_{i0}^0 \) - the initial intensity of the longitudinal and transverse waves.

### 4 Analysis of the behavior solutions near the fronts of waves

For the analysis of the behavior solutions near the fronts of the shock waves for velocities and strains of the dynamic problem of the propagation waves in elastic medium microstructure, it is necessary to implement the distension of space along the normal to surface \( \Sigma \) of the fronts of the shock waves. This distension
corresponds to distension in time, because the distance by normal behind front of
the passing wave is defined as \( n = G t \).

We will use the curvilinear frame of reference \((n, y_1, y_2, t)\). Besides, in
equation (7) we will change the differentiation \( \frac{\partial}{\partial t} \) and \( x \) - by differentiation
\( \frac{\delta}{\delta t} \) and \( \frac{\partial}{\partial n} \), \( \frac{\partial}{\partial y_1} \), \( \frac{\partial}{\partial y_2} \), where \( \frac{\delta}{\delta t} \) - local partial derivative
by time for function \( f(n, y_1, y_2, t) \), which is defined in movable axis.

It is convenient to decrease the order of the system (7) by the introduction of the
additional variables:
\[
W_k = u_{k, kk} \quad Q_i = u_{i, kk} .
\]
Then the equations of motion (7) get the form

\[
\rho u_{i,tt} = \lambda \{ u_{k, kk} + 2\alpha u_{k, k} (u_{k, k}) + \beta \frac{\mu}{\lambda} u_{k, i} (u_{k, li} + u_{li, k}) \} + \\
+ 2\mu \frac{1}{2} (u_{l, li} + u_{l, ij}) + u_{i, k} (u_{k, li} + u_{li, k}) + u_{i, l} (u_{l, li} + u_{li, l}) \}
+ \lambda \frac{h^2}{12} \{2W_{k, kk} + 4\alpha u_{i, li} W_{k, k} + 4\alpha u_{i, l} W_{k, i} + \\
+ \beta \frac{\mu}{\lambda} ((u_{k, li} + u_{k, ik})[Q_{k, li} + Q_{l, ki}] + (u_{l, li} + u_{l, ik})[Q_{k, li} + Q_{l, ki}] + \\
+ 2u_{h, k} (Q_{i, li} + Q_{l, ki}) + u_{k, k} (Q_{i, li} + Q_{l, ki}) + u_{k, l} (Q_{i, li} + Q_{l, ki}) + \\
+ [W_{k, k} (u_{i, li} + u_{j, ij}) + W_{k, i} (u_{i, li} + u_{j, ij}) + \nu \frac{1}{2} (Q_{l, k} + Q_{k, li}) (u_{k, l} + u_{j, l}) + \\
+ (Q_{l, k} + Q_{k, li}) (u_{k, li} + u_{j, ij}) + (u_{l, ki} + u_{k, li}) (Q_{k, li} + Q_{l, ki}) + \\
+ (u_{i, l} + u_{k, ki}) (Q_{k, li} + Q_{l, ki}) \} \}
\]

To investigate the longitudinal waves, which spread in linear approximation with
the velocity \( \rho G^2 = \lambda + 2\mu \), we project the equation of motion (12) to the
normal.

We will introduce the small parameter \( \varepsilon \) for the system of equations of motion:
\[
\alpha = \varepsilon \alpha^* \quad \beta = \varepsilon \beta^* \quad \nu = \varepsilon \nu^* \quad \text{where} \quad \alpha^*, \beta^*, \nu^* - \text{the}
\]
constants of the material, and \( \varepsilon \) - order of small of this constant. Thus, the
equations of motion of the longitudinal and shear waves will include the small
values \( \varepsilon \) and \( h \).

The small parameter \( \delta \) also will be introduced on the condition:
\( \delta^2 = h^2 / \varepsilon \).

After performing the procedure of distension in time \( t = 2T / \varepsilon \), retaining the
terms with order \( \delta^2 \), the system of equations of motion for the longitudinal
wave gets the form
\[-2\rho G u_{n,nT} = 4\lambda\alpha^*(u_{n,mn}u_{n,n} - 2\Omega u_{n,n}^2 - 2\Omega u_{n,mn}u_{n} - 8\Omega^2 u_{n,n}u_{n} - 8\Omega^3 u_{n,n}^2 + 2(3u_{n,mn}u_{n,n} + \mu u_{n,mn}u_{n,n} - 4\Omega u_{n,mn}u_{n} - 2\Omega u_{n,mn}u_{n} - 4\Omega^2 u_{n,n}^2 + 12\Omega^2 u_{n,n}u_{n} + 16\Omega^3 u_{n,n}^2 + 8\Omega^3 u_{n,n}) + \frac{\lambda}{6}\delta^2(2\frac{h}{h^2}u_{n,mn} - 4\Omega u_{n,mn} + 8\Omega^3 u_{n,n}).\]  

For the case of small tangent displacement \(u_a\) on the front of longitudinal wave \(\Sigma (u_a \approx 0)\), the equation in projection to the normal \(n\) is reduced to the equation for one normal component \(u_n\)

\[-2\rho G u_{n,nT} = 4\lambda\alpha^*(u_{n,mn}u_{n,n} - 2\Omega u_{n,n}^2 - 2\Omega u_{n,mn}u_{n} - 8\Omega^2 u_{n,n}u_{n} - 8\Omega^3 u_{n,n}^2 + 2(3u_{n,mn}u_{n,n} + \mu u_{n,mn}u_{n,n} - 4\Omega u_{n,mn}u_{n} - 2\Omega u_{n,mn}u_{n} - 4\Omega^2 u_{n,n}^2 + 12\Omega^2 u_{n,n}u_{n} + 16\Omega^3 u_{n,n}^2 + 8\Omega^3 u_{n,n}^2) + \frac{\lambda}{6}\delta^2(2\frac{h}{h^2}u_{n,mn} - 4\Omega u_{n,mn} + 8\Omega^3 u_{n,n}).\]  

It is impossible to build an analytic solution for the equation (13).

For the case of wave fronts with a small curve \(\Omega \to 0\) the equation (14) is reduced to the one-dimensional equation of Korteweg-de Vries [3].

\[u_{n,nT} + \Lambda u_{n,mn}u_{n,n} + \delta_0^2 u_{n,mnm} = 0,\]  

where \(\Lambda = \frac{2\lambda\alpha^* + 3\mu\beta^* + 2\mu\nu^*}{\rho G}\), \(\delta_0^2 = \frac{\lambda}{6\rho G}\).

For the volumetric deformation \(e = e_{n,n}\) the equation (15) is reduced

\[e_{,T} + \Lambda ee_{,n} + \delta_0^2 e_{,mn} = 0.\]  

Transforming the measure by \(n, n = \Lambda \cdot N\), we get [3]

\[e_{,T} + ee_{,n} + \delta^* e_{,NNN} = 0,\]  

where \(\delta^* = \frac{\delta^2}{6(2\lambda\alpha^* + 3\mu\beta^* + 2\mu\nu^*)}\).

The equation of Korteweg-de Vries (17) for the volumetric deformation on the front of the plane longitudinal wave corresponds to solution [3]
where $k^2 = \text{const}$ - amplitude of wave, $N_0$ - the initial position of wave.

Thus, the solution for the volumetric deformation in the transitional area of the plane nonlinear elastic wave presents the isolated wave of constant amplitude. Regarding the longitudinal wave, the velocity is proportional to the amplitude.

The solution in the form of the isolated wave can be used for the conducted amalgamation of external decompositions in the transitional area of the longitudinal wave. Consequently, the structure of the transitional area of the plane longitudinal wave is not stationary.

The absence of the pointing analytical solution for the volumetric deformations in the transitional area near the wave of the surface $\Sigma$ with arbitrary geometry does not allow us to get the exact distribution of the deformation in the transitional area.

However, the knowledge of solution for deformation behind plane-isolated waves allows us to do some qualitative conclusion for waves on the unfolding and refolding space surface $\Sigma$.

For the case of unfolding longitudinal wave $\Omega_0 < 0$ the external decomposition gives attenuation of intensity of the longitudinal waves in the process of their propagating. Consequently, their amplitude $k^2$ decreases in the process of spreading. Thus, the velocity of the spread of the isolated wave in the transitional area is slowed down.

For the case of refolding longitudinal waves, the amplitude and velocity of the isolated wave in the transitional area increases.

References