Dual reciprocity method for elastodynamics in infinite domains

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Abstract

This paper presents a novel dynamic modeling of structures in infinite domains. The dual reciprocity method (DRM) using compact supported radial basis functions (CS RBF) is used. Four different functions are considered. The present formulation is general, as it proved validity for modeling any type of structures including structures in infinite domains. An example of deep tunnel is analyzed to demonstrate the validity of the present formulation.

1 Introduction

Structures in infinite domains such as deep tunnels are usually subjected to dynamic loading, such as earthquakes, traffic loading, machine loading or even explosions. In this decade the development of supercomputers together with the new advances in the numerical solution of partial differential equations make it possible to analyze complex structures under such loading. The most powerful numerical technique used by the industry nowadays is the Finite Element Method (FEM) [1]. However, FEM requires domain discretisation and is not very suitable for problems involving stress concentrations or in computing high frequency modes as well as it needs special elements to model infinite domains. The Boundary Element Method (BEM) [2], on the other hand, has made it possible to solve such problems using boundary-only discretisations. This leads not only to a decrease in both modeling requirements and human effort, but also to a considerable improvement in the numerical accuracy.

The development of BEM for dynamic problems has been a subject of research since the 1970s. In the early 1980s, there was a major development in the BEM known as the Dual Reciprocity Method (DRM) developed by Nardini and Brebbia [3]. The DRM has been used to solve potential, fluid mechanics and heat transfer problems [4], but there has been very few publications applying the method to dynamic problems. Among these publications are the work of Bridges and Wrobel [5] where spline functions is used to model free vibration problems. Agnantiaris et al. [6] compared the behaviour of polynomial RBF against thin
method to dynamic problems. Among these publications are the work of Bridges and Wrobel [5] where spline functions is used to model free vibration problems. Agnantiaris et al. [6] compared the behaviour of polynomial RBF against thin plate splines. Rashed [7] extended the DRM to dynamics using Gaussian functions. Agnantiaris et al. [8] recently used the same formulation with polynomial and multi-quadratic functions 3D and axisymmetric structures. The formulation also applied to non-linear applications by Telles and Carrer [9] and by Coda and Venturini [10]. Extension of the formulation to the Galerkin-type collocation is studied by Perez-Gavilian and Aliabadi [11]. Additional work for soil-structure interaction and fracture mechanics can be found in the textbook of Dominguez [12]. It has to be noted that non of the formerly mentioned formulation supports the analysis of structures in infinite domains.

The main idea of the DRM is to treat the inertia terms as pseudo body forces. Then writing another integral representation of these pseudo body forces. Both the original and the body force integral formulation use the Green reciprocity theory; therefore the formulation is said to be dual reciprocity formulation. In order to represent the inertial terms as body forces, two important steps are carried out:

1- The new body forces should be prescribed everywhere; therefore the acceleration term has to be interpreted in terms of the displacements. This could be done using finite difference scheme in time [1,12].

2- A collocation process should be carried out to represent the pseudo body forces as an interpolated continuous surface. This step could be done using Radial Bases Functions (RBF).

Most of the DRM development until recently uses the ad-hoc \((1+R)\) basis function in the approximation of the inertia terms. Golberg and Chen [13] studied the different types of RBF. They showed that the \((1+R)\) function is just one type of a class of RBF called conical. They presented also alternative classes of RBF, such as splines, multi-quadratic, etc. Some of these functions were used in dynamic analysis as mentioned earlier. Despite of the many special features of RBF, it is known that most of the RBF are globally defined basis functions. This means the resulting matrix for interpolation is dense and can be highly ill conditioned, especially for a large number of interpolation points. This poses serious stability problems and high computational cost. All of these drawbacks lead the researchers to search for basis functions that have local support. The most popular compact supported RBF (CS RBF) are the ones constructed by Wendland [14]. Chen et al. [15] implemented the CS RBF with the DRM to solve potential problems. Golberg et al. [16] used similar formulation for 3D problems and Cheng et al. [17] studied the performance of CS RBF with iterative solvers. Rashed [18] derived the particular solutions corresponding to CS RBF for elasticity equations.

One of the advantages of the BEM is the modeling of external domains. In this case only the internal surface of the structure is needed to be discretised. The regularity conditions are satisfied at the infinity [2]. Loeffler and Mansur [19] were the first to try the application of the DRM to infinite domain potential problems. They used special class of RBF to guarantee the decaying at infinity.
Zhu and Zhang [20] proposed an alternative approach based on mathematical mapping. It has to be noted that such formulations are not suitable for elasticity equations due to its complexity. By inspection of the behavior of the present CS RBF, it can be easily seen that they decay outside the compact zone. Moreover, due to the nature of the infinite domain modeling, increasing the compact radius will not have any effect on the solution as the problem is extended to infinity.

In this paper, the application of the dual reciprocity method (DRM) to the dynamic analysis of elasticity problems is extended using CS RBF. A deep tunnel example is presented to demonstrate the accuracy and the validity of the proposed formulation.

2 Dual reciprocity formulation

For a general body having a domain $\Omega$ with boundary $\Gamma$, if $u_j$ & $t_j$ denote the boundary displacement & traction vectors respectively and $U_j$ denote the acceleration term, the corresponding boundary integral equation can be defined as follows [12]:

$$C_{ij}u_j + \int_{\Gamma} T_{ij}u_j d\Gamma - \int_{\Gamma} U_{ij}t_j d\Gamma = \int_{\Omega} U_{ij} \ddot{u}_j d\Omega$$

(1)

where, $C_{ij}$ denotes the jump terms. The $U_{ij}$ & $T_{ij}$ denote the two-point fundamental solution kernels. If the same problem is re-defined using a collocation based field of displacements and tractions $\Psi_{jm}$ and $\eta_{jm}$ respectively, Eq. (1) can be redefined as follows:

$$\left(C_{ij} \Psi_{jm} + \int_{\Gamma^*} T_{ij} \Psi_{jm} d\Gamma^* - \int_{\Gamma^*} U_{ij} \eta_{jm} d\Gamma^*\right) \alpha_m = \int_{\Omega^*} U_{ij} \ddot{U}_j d\Omega^*$$

(2)

where the domain $\Omega^*$ and its boundary $\Gamma^*$ can be chosen arbitrary and $\ddot{U}_j$ is the new body acceleration term that generates the displacements and tractions $\Psi_{jm}$ and $\eta_{jm}$, and $\alpha_m$ is the coefficient of collocation which can be defined as:

$$\ddot{U}_j = f \alpha_j$$

(3)

where $f$ can be chosen as arbitrary function. If the $\Omega^*$ and $\Gamma^*$ are chosen to be $\Omega$ and $\Gamma$ and $\alpha_j$ is computed assuming that $\ddot{U}_j$ is equal to $\ddot{u}_j$, one can use both
Eqs (1) and (2) to solve the problem with boundary-only integrals. Eqs (1) and (2) which are based on the Green's reciprocal theory, are the basic equations for the dual reciprocity formulation.

Two main steps remain to be performed before setting up the final formulation. The first is to interpolate $\ddot{u}_j$ in terms of $u_j$ in order to generate prescribed L.H.S. for Eq. (3). This can be done using a suitable finite difference scheme [1]:

$$
\dot{u}_j = \frac{2\langle u_j \rangle^{t+\Delta t} - 5\langle u_j \rangle^t + 4\langle u_j \rangle^{t-\Delta t} - \langle u_j \rangle^{t-2\Delta t}}{\Delta t^2}
$$

(4)

Where $\langle u_j \rangle^t$ is the value for the displacements at time $t$ and $\Delta t$ is an appropriate time step. Then the final formulation can be obtained as follows [7]:

$$
C_{ij}\langle u_j \rangle^{t+\Delta t} + \int_{\Gamma} T_{ij} \langle u_j \rangle^{t+\Delta t} \, d\Gamma = \int_{\Gamma} U_{ij} \langle t_j \rangle^{t+\Delta t} \, d\Gamma
$$

$$
+ \left[ C_{ij} \Psi_{ij} + \int_{\Gamma} T_{ij} \Psi_{ij} \, d\Gamma - \int_{\Gamma} U_{ij} \eta_{ij} \, d\Gamma \right]
$$

$$
\times \frac{\rho}{(\Delta t)^2} \left[ 2\langle u_i \rangle^{t+\Delta t} - 5\langle u_i \rangle' + 4\langle u_i \rangle^{t-\Delta t} - \langle u_i \rangle^{t-2\Delta t} \right]
$$

(5)

where, $\rho$ denotes the body density. The second step is to compute the expressions for the fictitious displacements and tractions. It has been proven by Rashed [7] that such fields can be expressed in terms of the fictitious Galerkin tensor $g$ as follows:

$$
\Psi_{ij} = \frac{1}{\mu} \delta_{ij} \left[ \frac{g'}{R} + g'' \right] - \frac{1}{2(1-\nu)\mu} \left[ \frac{g'}{R} \left( \delta_{ij} - R_{ri} R_{sj} \right) + g'' R_{ri} R_{sj} \right]
$$

(6)

and
where \( \eta \) denotes the shear modulus and \( \nu \) denotes the Poisson's ratio, \( R \) is the Euclidian distance between the field and the source points and 

\[

\eta_{ij} = N_i R_{ij} \left[ \frac{\nu}{1-\nu} \left( \frac{g'''}{R^2} - \frac{g''''}{R} \right) + g''' \right] \\
+ N_i R_{ij} \left[ \left( \frac{g'}{R^2} - \frac{g''}{R} \right) + \frac{\nu}{1-\nu} g'''' \right] + \\
\delta_{ij} R_{ij} \left[ \frac{\nu}{1-\nu} \left( \frac{g'}{R^2} - \frac{g''}{R} \right) + g'''' \right] - R_{ij} R_{ij} \frac{1}{1-\nu} \left[ 3 \left( \frac{g'}{R^2} - \frac{g''}{R} \right) + g''' \right]
\]

\[

(7)
\]

where \( \mu \) denotes the shear modulus and \( \nu \) denotes the Poisson's ratio, \( R \) is the Euclidian distance between the field and the source points and 

\[

\nabla^4 g = f 
\]

in which \( \nabla^4 \) is the two dimensional bi-harmonic operator. If the function \( f \) is chosen to be one of the CS RBF in given in Table 1 (see Wendland [6]), the expression for the fictitious displacements can be obtained from Ref. [18]. It has to be noted that the symbol (Expression)+ which used in Table 1 denotes: 

- if \( R < \alpha \) and \( f = 0 \); 
- if \( R > \alpha \), and \( \alpha \) denotes the radius of the compact support.

### 3 Deep tunnel example

The deep tunnel shown in Fig. 1 is considered. The results are presented for the four functions. The tunnel of radius \( \alpha \), and the following properties are considered for the surrounding soil: \( E = 650 \times 10^6 \) Pa, \( \nu = 0.2308 \) and \( \rho = 1800 \) Kg/m\(^2\). The applied load is taken equal to: \( F(t) = P \times H(t-0) \); where is \( H(t-0) \) is the Heaviside function. Quadratic boundary elements are used and the numerical integrations are carried out using four Gauss points. Only the boundary of the tunnel is needed to be discretised (in the clockwise direction to represent the infinite domain; see Fig. 1). Internal points are placed uniformly in the radial direction at offset distance equal to multiples of 0.5 \( \alpha \).

The results for the normalized radial displacement history for the four considered functions are given in Figs. 2-5 respectively (where \( c \) is the wave velocity [9]). The results are plotted against the results obtained from Ref. [9], where one quarter of the problem is considered as interior problem enclosed by fictitious boundaries. The compact support radius (\( \alpha \)) is fixed to be 2 \( \alpha \). It was found that results for smaller radius are not accurate. Also, the increasing of this radius more than 2 \( \alpha \) has a slight effect on the accuracy. It has to be noted that, in the case of using small number of the internal inertia nodes (24 int. points), the
accuracy greatly improved when using \( a = 4 \alpha \); therefore the shown results for the 24 int. points are obtained with radius of compact \( a = 4 \alpha \). The results using the Gaussian function is also given in Fig. 2.

Table 1: The used CS RBF.

<table>
<thead>
<tr>
<th>Number</th>
<th>Function</th>
<th>Continuity</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>( f = \left(1 - \frac{R}{\alpha}\right)^2 )</td>
<td>( C^{(0)} )</td>
</tr>
<tr>
<td>Second</td>
<td>( f = \left(1 - \frac{R}{\alpha}\right)^4 \left(1 + 4\frac{R}{\alpha}\right) )</td>
<td>( C^{(2)} )</td>
</tr>
<tr>
<td>Third</td>
<td>( f = \left(1 - \frac{R}{\alpha}\right)^6 \left(3 + 18\frac{R}{\alpha} + 35\left(\frac{R}{\alpha}\right)^2\right) )</td>
<td>( C^{(4)} )</td>
</tr>
<tr>
<td>Fourth</td>
<td>( f = \left(1 - \frac{R}{\alpha}\right)^8 \left(1 + 8\frac{R}{\alpha} + 25\left(\frac{R}{\alpha}\right)^2 + 32\left(\frac{R}{\alpha}\right)^3\right) )</td>
<td>( C^{(6)} )</td>
</tr>
</tbody>
</table>

Fig. 1: Deep tunnel.

Figures 2-5 show the good agreement between the present results with the results of Ref. [9]. It can be seen that by increasing the internal inertia nodes the result
Fig. 2: Radial displacement history for the deep tunnel problem: Results for the first function.

Fig. 3: Radial displacement history for the deep tunnel problem: Results for the second function.
Fig. 4: Radial displacement history for the deep tunnel problem: Results for the third function.

Fig. 5: Radial displacement history for the deep tunnel problem: Results for the fourth function.
improves. All four functions are accurate and valid to model such classes of problems. It has to be noted that, by increasing the number of the internal points, the first function has a fast convergence rate. The Gaussian function, on the other hand, gives divergent results for this type of problem.

4 Conclusions

The CS RBF has been successfully used in the implementation of the DRM for dynamics. The following conclusions might be drawn from the present work:

1- The use of the DRM allows the discretisation of the problem boundary only.
2- Using CS RBF is more reliable than using the Gaussian function as it is more stable and has convergence mathematical bases.
3- The CS RBF is suitable in modeling infinite domain problem.

As a recommendation for future work, investigations on using CS RBF with variable support radius might be significant. The present formulation could also be used in treatment of general body forces and for free vibration problems.

References


