Transactions on the Built Environment vol 26, © 1997 WIT Press, www.witpress.com, ISSN 1743-3509

# The Vibration of Masonry Pinnacles

J. Heyman
*3 Banhams Close, Cambridge, CB4 1HX, UK*

## Abstract

Slender structural elements often appear "lively" when scaffolded for repair: a masonry spire, for example, or a smaller tapering pinnacle, can be made to vibrate quite easily. This natural behaviour is examined by obtaining estimates of fundamental frequencies, using the Rayleigh-Ritz method.

## 1 Introduction

The type of pinnacle to be considered is shown schematically in fig. 1. The shape will be taken to be that of a right cone, whose cross-section may in practice be square or octangular. A real pinnacle will have crockets or other surface decoration, but examination of the conical shape gives results which lead to more or less general conclusions. (A more complex profile is considered later, fig. 5.)

Loading on the pinnacle will be taken to be lateral, whether due to horizontal wind forces or to inertia effects arising from vibration. Steady gravity loading from self-weight of the material will not enter the analysis, and the conventional engineering theory of bending will be applied to the "beam" of fig. 2. It will be seen from this figure that the cone does not come to a point; a frustum of length $\ell$ is being considered, of width $2a$ at the tip and $2b$ at the abutment. The termination at the abutment will be taken to be fully rigid, which leads to simpler calculations without too much loss of accuracy.

The reason for analysing initially a frustum rather than a full cone is apparent if, for example, the effect of a tip load $W$ is considered. The depth of the beam of fig. 2 at any section $x$ is $2(a + \gamma x/\ell)$, where $\gamma = (b - a)$, so that the second moment of area is

$$I_x = I_o\left(\frac{a}{b} + \frac{\gamma x}{b\ell}\right)^4 . \tag{1}$$

where $I_o$ is the value of $I_x$ at the abutment $x = \ell$. The differential equation of bending under the tip load $W$ is

$$EI_x \frac{d^2y}{dx^2} = Wx, \tag{2}$$

so that, using the conditions $y = 0$ and $dy/dx = 0$ at $x = \ell$, the tip deflexion is determined as

$$\delta = \frac{W\ell^3}{3EI_o} \cdot \frac{b}{a}. \tag{3}$$

It is clear that, as the value of $a$ approaches zero (the pinnacle comes to a point), so the tip deflexion becomes infinite. Equally the section modulus of such a truly conical pinnacle will be inversely proportional to the cube of the distance $x$ from the apex, whereas the bending moment is still given by equation (2) – the stresses will also become indefinitely large as $x$ approaches zero. (This observation may be compared with the known fact [1] that tips of masonry spires on churches are prone to exhibit distress. Compare also the crack of a whip; the tapering tip of the lash accelerates through the sound barrier.)

It is also clear that the actual finite dimensions near the tip of a pinnacle will prevent the infinities arising in the analysis; further, simple calculations show that the effects are confined to the region near the tip. However the region near the tip remains critical because of the unilateral nature of the material. Masonry is weak under tension: gravity loading will provide the necessary compressive "prestress" at a metre or so from the tip, but it will be assumed that the topmost courses are (as is usual in practice) bonded firmly together with metal connectors able to resist tensile forces.

## 2 Static deflexions of a cantilever

In order to introduce the approximate methods used to estimate the natural frequencies of masonry pinnacles, the static deflexion of the cantilever of fig. 2 will first be studied under the action of a tip load $W$. If the beam has uniform section ($a = b$) of second moment of area $I_o$, then direct solution of the bending equation gives the tip deflexion as $\delta = W\ell^3/3EI_o$.

Instead of obtaining directly the exact solution, a deflected form $Y = Y(x)$ will be assumed for the cantilever. If then the energy equation is written,

$$W\delta = \int_0^\ell EI_x \left(\frac{d^2Y}{dx^2}\right)^2 dx, \tag{4}$$

where $I_x = I_o$ and $\delta$ is the value of $Y$ at $x = 0$. (Factors of $\frac{1}{2}$ have been omitted from both sides of equation (4)). That function of $Y$ is correct which gives the largest value of $\delta$ from equation (4). Suppose, for example, it is assumed that

$$Y = \delta\left(1 - \frac{x}{\ell}\right)^2. \tag{5}$$

This parabolic form of the deflexion satisfies the conditions $Y$ and $dY/dx$ both zero at the fixed end $x = \ell$. Substitution in equation (4) leads to $\delta = W\ell^3/4EI$.

As a second trial, $Y$ will be taken as

$$Y = \delta\left(1+\alpha\frac{x}{\ell}\right)\left(1-\frac{x}{\ell}\right)^2. \tag{6}$$

which again satisfies the boundary conditions, but now has a free parameter $\alpha$. Substitution into equation (4) gives

$$\delta = \frac{W\ell^3}{4EI_o}\left(\frac{1}{1-\alpha+\alpha^2}\right). \tag{7}$$

The deflexion $\delta$ is a maximum for $\alpha = \frac{1}{2}$, and the exact solution $\delta = W\ell^3/3EI_o$ is in fact obtained: the deflexion of the uniform cantilever is actually a cubic function of $x$, and is given exactly by equation (6) with $\alpha = \frac{1}{2}$.

These techniques will be applied to the tapering cantilever of fig. 3, i.e. a true cone with $a = 0$. Under a tip load $W$, the "exact" solution (obtained by integrating the differential equation, and ignoring the breakdown of the analysis as $x$ approaches zero), is given by

$$y = \frac{W\ell^3}{3EI_o}\left[\frac{3/x+3x-6}{2}\right]. \tag{8}$$

The approximate solutions, using equation (5) and (6) for the assumed deflexions, are

$$\text{Quadratic} \quad : \quad y = \frac{W\ell^3}{3EI_o}\cdot\frac{15}{4}\left(1-\frac{x}{\ell}\right)^2. \tag{9}$$

$$\text{Cubic} \quad : \quad y = \frac{W\ell^3}{3EI_o}\cdot\left(1-\frac{7}{6}\frac{x}{\ell}\right)\left(1-\frac{x}{\ell}\right)^2. \tag{10}$$

The three expressions, equations (8), (9) and (10), are plotted in fig. 4. It will be seen that the cubic expression is a good approximation to the actual behaviour for $x/\ell > 0.2$.

The analysis may be repeated (not reported here) for the frustum of fig. 2. The "exact" displacements remain finite, and, as might be expected, the best cubic expression lies close to the exact solution even for values of $a/b$ as small as 0.1.

## 3 Vibration of a conical pinnacle

Following Timoshenko [2], the differential equation for vibration of a beam of variable section $A_x$ and flexural rigidity $EI_x$ is given by

$$\frac{\partial^2}{\partial x^2}\left(EI_x\frac{\partial^2 y}{\partial x^2}\right) + A_x\rho\frac{\partial^2 y}{\partial t^2} = 0. \tag{11}$$

where $\rho$ is the unit mass of the material. Unless the cross-section varies in some particular way leading to easy integration, it is in general not easy to find closed-form solutions of equation (11). Instead, approximate methods, such as the Rayleigh-Ritz method, may be used to calculate the fundamental natural frequency of vibration. The deflexion of the beam is taken in the form

$$y = Y \sin \omega t, \tag{12}$$

where the function $Y$ gives the shape of the vibration. Then the maximum potential (bending) and kinetic energies may be written

$$\left.\begin{aligned} V &= \tfrac{1}{2}\int_0^\ell EI_x \left(\frac{d^2Y}{dx^2}\right)^2 dx, \\ \text{and} \quad T &= \tfrac{1}{2}\omega^2 \int_0^\ell A_x \rho Y^2 dx. \end{aligned}\right\} \tag{13}$$

If these two expressions are equated, then

$$\omega^2 \int_0^\ell A_x Y^2 dx = \frac{E}{\rho}\int_0^\ell I_x \left(\frac{d^2Y}{dx^2}\right)^2 dx. \tag{14}$$

The true expression for the mode shape, $Y$, will lead to a minimum value of $\omega$ from equation (14).

For the conical pinnacle of fig. 3 the value of $A_x$ is given by $A_o x^2/\ell^2$, and, as before, $I_x = I_o x^4/\ell^4$. Thus, for any reasonable function which may be assumed for the deflected shape $Y$, the loading boundary conditions at the end $x = 0$ will be satisfied, that is, both bending moment and shear force will be zero:

$$\left(EI_x \frac{d^2Y}{dx^2}\right)_{x=0} = 0 \quad \text{and} \quad \frac{d}{dx}\left(EI_x \frac{d^2Y}{dx^2}\right)_{x=0} = 0. \tag{15}$$

The simplest function for $Y$ that satisfies the displacement boundary conditions at $x = \ell$ (slope and deflexion both zero) is given by the parabolic expression, equation (5). Substitution into equation (14) gives

$$\omega^2 = (84)\left(\frac{E}{\rho} \cdot \frac{I_o}{A_o \ell^4}\right). \tag{16}$$

This value for $\omega$ may be improved by taking for $Y$ the cubic function of equation (6). Substitution into equation (14) gives

$$\omega^2 = (144)\left(\frac{7 + 7\alpha + 3\alpha^2}{12 + 9\alpha + 2\alpha^2}\right)\left(\frac{E}{\rho}\frac{I_o}{A_o \ell^4}\right). \tag{17}$$

The value of $\omega$ is minimised for $\alpha = -0.575$, leading to

$$\omega^2 = (76.30)\left(\frac{E}{\rho}\frac{I_o}{A_o \ell^4}\right). \tag{18}$$

Transactions on the Built Environment vol 26, © 1997 WIT Press, www.witpress.com, ISSN 1743-3509


Some idea of the accuracy of these calculations may be obtained by applying the techniques to the vibration of a cantilever of uniform section. The simplest quadratic function gives the coefficient for $\omega^2$ (cf. 84 in equation (16)) as 20, while the cubic gives the value as 12.48. The uniform cantilever may in fact be solved exactly [2], and the correct value of the coefficient for $\omega^2$ is 12.36.

# 4 A realistic pinnacle

In practice, a pinnacle may have a shape such as that idealised in fig. 5, namely a cylinder (of square or octagonal section) surmounted by a cone. If the Rayleigh-Ritz analysis is repeated for the simple quadratic form for $Y$, equation (5), then

$$\omega^2 = 84\left(\frac{E}{\rho}\cdot\frac{I_o}{A_o\ell^4}\right)\left[\frac{\beta^2(5-4\beta)}{1-(1+6\beta)(1-\beta)^6}\right]. \qquad (19)$$

It will be seen that for $\beta = 1$ (the full conical pinnacle) the solution of equation (16) is recovered; for $\beta = 0$ (the pinnacle of uniform section), the bracketed term in equation (19) has value 5/21, and the coefficient of 20 for $\omega^2$ is recovered. For $\beta = \frac{1}{2}$, the coefficient for $\omega^2$ is 67.2.

The cubic mode for $Y$, equation (6), leads to heavy algebra, although the integrations are straightforward. For $\beta = \frac{1}{2}$, the minimum value of $\omega$ occurs for $\alpha = 1.042$, and the corresponding coefficient for $\omega^2$ is 53.24.

The frequency of vibration is $\omega/2\pi$, and may therefore be expressed as

$$f = \frac{\omega}{2\pi} = \frac{k}{\ell^2}\sqrt{\frac{E}{\rho}\cdot\frac{I_o}{A_o}}, \qquad (20)$$

where the value of $k$ may be read from the following table, incorporating the results obtained above:

Table: Values of $k$, equation (20)

| | | | Mode 1 eqn (5) | Mode 2 eqn (6) | Exact |
|---|---|---|---|---|---|
| Uniform section | : | $\beta = 0$ | 0.7118 | 0.5622 | 0.559 |
| Pinnacle, fig. 5 | : | $\beta = \frac{1}{2}$ | 1.305 | 1.161 | – |
| Cone, fig. 1 | : | $\beta = 1$ | 1.459 | 1.390 | – |

# 5 Pinnacles at Ely and King's College Chapel, Cambridge

After the fall of the central tower at Ely Cathedral in 1322, the crossing was opened out into an octagon; the flying buttresses helping to support the new masonry are stabilized at their ends by four massive pinnacles, each of about 20 tonnes mass. The cross-section of a pinnacle is approximately

Transactions on the Built Environment vol 26, © 1997 WIT Press, www.witpress.com, ISSN 1743-3509


square (of side $2b$), so that the ratio $I_o/A_o$ is $b^2/3$, and equation (20) may be rewritten

$$f = k\frac{b}{\ell^2}\sqrt{\frac{E}{3\rho}}. \tag{21}$$

The length $\ell$ is about 11 m, and the width $2b$ about 1 m. The value of $\beta$ (fig. 5) is about $\frac{1}{2}$. The unit mass $\rho$ is about 2000 kg/m$^3$, but Young's Modulus $E$ cannot be estimated so easily. Stone is a variable material, and the mortar beds may have moduli one or two orders of magnitude different from that of the stone itself. The value of 20 kN/mm$^2$ will be used; it may be noted that the frequency of vibration is proportional to the square root of the modulus. Using these values, the fundamental frequency of vibration of the pinnacles at Ely may be estimated from equation (21), using $k = 1.161$, as 8.8 Hz.

The south side of the octagon was scaffolded in June 1996, enabling access to two pinnacles. Dr H.E.M. Hunt of the Department of Engineering, University of Cambridge, attached his sophisticated and highly portable vibration apparatus to these pinnacles, and direct measurements of frequency were made. (The apparatus consists of a notebook computer fitted with a data-acquisition card for collecting data from two accelerometers.) The south-east pinnacle had fundamental frequencies of 4.5 and 5.6 Hz in the NE-SW and SE-NW directions respectively; similar results were obtained for the other pinnacle. The cross-section of a pinnacle is in fact not square, but slightly rhomboid, and each of the four faces is cut back to give a panelled surface.

At King's College Chapel twenty-two pinnacles surmount the parapet; each is of length (to the point where the section increases sharply) of about 7.25 m and of width $2b = 0.95$ m. Again $\beta$ is about $\frac{1}{2}$, and the same material constants give a fundamental frequency of 19 Hz. The tapering portion of each pinnacle is actually rather slight, not starting from the full width of 0.95 m; if this portion is ignored altogether, and the length taken as 4 m of uniform section with $k = 0.559$ from the table, then the frequency is determined as 30 Hz.

## 6 Arcade columns

A common observation is that a free-standing column "sings" when struck with the palm of the hand or a soft hammer – a lowish note may be heard. Such columns occur, for example, in the inner arcades of some towers, or at triforium level. The frequency of vibration is given by an expression of the same form as equation (20); for a column with pinned ends,

$$f_1 = \frac{\pi}{2\ell^2}\sqrt{\frac{E}{\rho}\cdot\frac{I}{A}}, \tag{22}$$

while, if the ends are fixed, the frequency becomes $f_2 = 2.27 f_1$.

Transactions on the Built Environment vol 26, © 1997 WIT Press, www.witpress.com, ISSN 1743-3509


Thus, for a column of diameter 200 mm and length 4 m, and using the same material constants as before, $f_1$ is determined as about 16 Hz, and $f_2$ as 35 Hz.

It may be noted that the buckling stress $\sigma_E$ of this column is 31 N/mm$^2$ if the ends are pinned, and 123 N/mm$^2$ if they are fixed. These figures are very high compared with the stress $\sigma$ actually carried by a typical column: the frequency of vibration under mean stress $\sigma$ is reduced in the proportion $(1 - \sigma/\sigma_E)^{\frac{1}{2}}$. This reduction will therefore be small, and no information will be obtained about the state of stress in the column merely by determining the note emitted.

## 7 References


1. Heyman, J. Hemingbrough Spire, in *Structural repair and maintenance of historical buildings II*, **1**, *General studies, materials and analysis* (ed. C.A. Brebbia, J. Dominguez, F. Escrig), Southampton (Computational Mechanics Publications), 1991, p.13.

2. Timoshenko, S. *Vibration problems in engineering*, New York (Van Nostrand), 1928, and later editions.


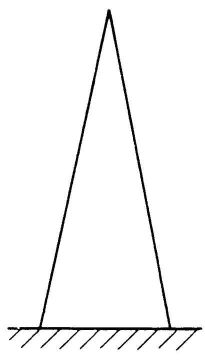

FIG.1

Transactions on the Built Environment vol 26, © 1997 WIT Press, www.witpress.com, ISSN 1743-3509


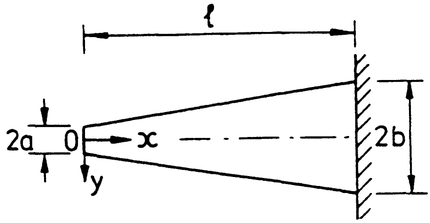


FIG. 2

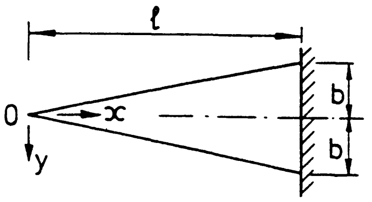


FIG. 3

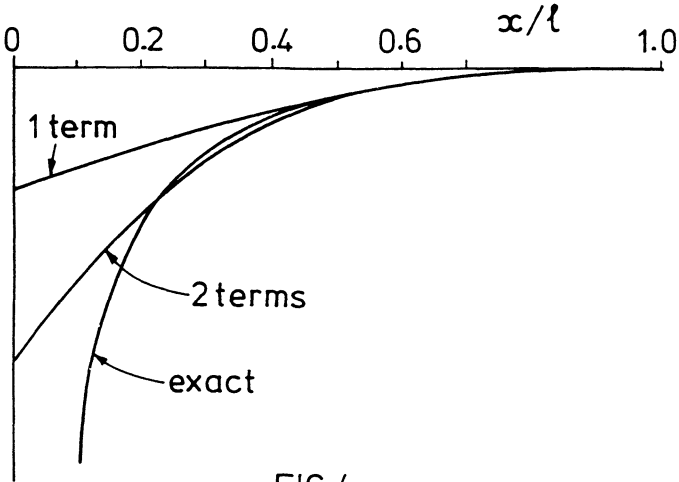


FIG. 4

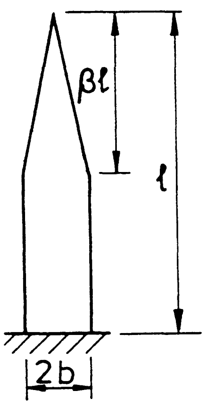


FIG. 5