On lines of thrust and stability of masonry arches
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Abstract

The use of lines of thrust to assess the stability of masonry arches, has become very popular on last years. Nevertheless, this form of analysis can be not sure in some cases. The real requirement for the stability of arches is that eccentricity of normal forces lies into the depth of the arch. It has been proposed that the same is true for lines of thrust. This paper shows that in many instances the latter is not the necessary condition.

1. The general principles of the stability of masonry

As it is well known, a cracked masonry structure can support a bending moment \( M \) if an axial force \( N \) acts simultaneously. As it can be shown in figure 1, compressions due to normal force form a pair with this force, \( N \cdot d \) that counteracts bending moment.
As \( d = \frac{h}{2} - \frac{h_0}{2} \), if we call \( h \) the depth of the structure and \( h_0 \) that of the compressed zone; we can express the equilibrium as:

\[
M = N \cdot d = \frac{N}{2} (h - h_0)
\]

and obtain:

\[
h = h_0 + \frac{2M}{N}
\]

as \( \frac{M}{N} = d \), where \( d \) is eccentricity of axial force and \( h_0 \), the depth strictly necessary to resist axial force without bending. Thus, the condition of stability of a masonry structure becomes (Fig. 2):

\[
h \geq h_0 + 2d
\]

If \( h \leq 2d = \frac{2M}{N} \) an arch is not stable, which does not depend on the values of \( M \) or \( N \) alone.

We can then deduce that the crucial value for the stability of an arch is the eccentricity of axial force \( d = \frac{M}{N} \). The value of \( h_0 \) is frequently negligible, as masonry structures have usually very low stresses. Based on the lower bound theorem of limit analysis, Heyman (1) has shown
that if a state of equilibrium with loads can be found, in such a way that the condition of
stability is satisfied along the structure, the arch is sure. This gives a simple and excellent
method for obtaining the values of bending moments and axial forces -and so the values of $d$-
along the arch. The problems arise when the value of $d$ is obtained by geometrical methods, as
it is shown hereafter.

2. The relationship between excentricity of normal forces and the
forces polygon

Suppose that we obtain by means of the forces polygon the funicular of the loads that
act at an arch, and that this funicular has the same thrust and vertical reactions of that arch. If
we draw the inverted funicular along the arch axis, we can obtain geometrically the values of $y = \frac{M}{H}$

and $d = \frac{M}{N}$, in the following way.

Giving a vertical section that includes the arch and the funicular (Fig. 3), and taking
moments around the arch axis, we obtain $M - H \cdot y = 0$, considering that the horizontal
component of the funicular normal force $N_y$ is $H$. We have then $y = \frac{M}{H}$, that is, the vertical
distance $y$ between the funicular of loads -"line of thrust"- and the axis of the arch, is the
excentricity of the thrust. This gives a graphical method to obtain the bending moments diagram
of the arch.
The excentricity of axial force, \( d \), can also be obtained graphically at the same section (Fig. 4). If the axial force of the funicular, \( N_a \), is slided along its line of action until it crosses the perpendicular line to the arch axis at the section, we can take moments around the axis of the arch and obtain \( M - N \cdot d = 0 \).

As \( N_a \) is the resultant of all forces acting at the left of the vertical section, its component in the arch axis direction is directly the axial force \( N \) of the arch. We obtain \( d = \frac{M}{N} \), that is, \( d \) is the distance between the tangent line of the funicular and the arch axis measured perpendicular to this axis. Two important remarks must be pointed out:

a) The distance \( d \) is not always shorter than the distance \( y \). As it can be seen in figure 5, if we apply the synus theorem, we obtain:

\[
\frac{y}{\sin\left[90 - \left(\gamma - \gamma_a\right)\right]} = \frac{d}{\sin\left(90 - \gamma_a\right)}
\]
calling $\gamma$ the slope of the arch, and $\gamma_0$ that of the funicular. We can then obtain:

$$d = y \frac{\cos \gamma_a}{\cos(\gamma - \gamma_a)}$$

We can easily deduce that if $\gamma_a < \frac{\gamma}{2}$, then $d > y$ and this difference can be very high if $\gamma_a << \frac{\gamma}{2}$.

b) The line that evolves all the distances "d", (line of pression), is only the funicular of forces if this line is a straight one. It is not the same line if the funicular of forces or "line of thrust" has changes in its slope, or is in general a curved line. As it has been previously shown the extremes of the distances $d$ are not the perpendicular distances from the axis of the arch to the line of thrust but the perpendicular distances from the axis of the arch to the tangent of the line of thrust at the vertical section.

One simple example may illustrate these remarks.

3. The triangular arch with two loads

Let us study a two-hinged triangular arch, submitted to two vertical forces $P$.

If we obtain a value of $H$ such as that of figure 6, bendings moments $M$ and normal forces $N$ can be readily obtained, as well as diagrams of $y$ and $d$. It can be observed that $y$ diagram is the line of thrust, but $d$ diagram is only the line of thrust in part of the arch. At the changes of slope of line of thrust -that correspond to the changes of the $N$ value- $d$ diagram continues as a horizontal line, until it intersects the perpendicular to the arch at the point of application of $P$. 

![Diagram of a triangular arch with two loads](image-url)
Line of pression looks more as a diagram than as a line of forces. If we try to evaluate the depth \( h \) of the arch, as that containing the line of thrust, we are bound to commit a great mistake at point A (Fig. 7), as the condition of stability demands an arch's depth three times greater.

4. **Continuous loads**

As we have seen, lines of thrust are not the same as lines of pression, for continuous loads. It can be deduced of lines of thrust applying the relation 

\[
d = y \frac{\cos \gamma_a}{\cos (\gamma - \gamma_a)}
\]

obtained above. For vertical loads (Fig. 8), the difference between these two lines is scarcely appreciable. Only if \( \gamma \) and \( \gamma_a \) are very different, the two curved lines have very different form, as in the case of figure 9, that corresponds to a wind load.

5. **Inclined loads. Ribs of gothic vaults**

For inclined loads, the value of the thrust \( H \) is also variable along the arch. That means that the thrust line is not proportional to the bending moments diagram. It presents discontinuities when the value of the thrust changes and so the funicular of forces represent the line of thrust only if this line is a straight one. As an example, we can see the triangular arch...
submitted now to two inclined loads (Fig. 10). A discontinuity can be appreciated at the point of application of \( P \) that compels the line of thrust to be far from the funicular of loads.

On the contrary (Fig. 11), the line of pression continues with a form very similar to that of vertical loads, and it continues to be evolved by the funicular of loads.

The discontinuities in the line of thrust has led many authors to draw the polygon of forces with vertical lines of action of loads, instead of inclined ones, as should be the correct way. But the fact is that the real discontinuities -those of \( d \)- can be greater than the ones obtained with that method (Fig. 12).

One example of that viewpoint is that of ribs of gothic vaults, studied by Mohrmann and Rosenberg (2). A gothic rib vault can be supposed as formed by separated arches supported at the ribs. Following Heyman's criterium, if the rib can support that forces, the structure is
sure. The arches that form the vault introduce inclined loads at the ribs. Rosenberg used line of thrust supposing that lines of action of loads are vertical, to state the stability of ribs. The results seems satisfactory (Fig. 13), but the fact is that, if we draw the line of pression, there appear five points in which the condition of stability is violated (Fig. 14).

Gothic ribs seems then scarcely sure studied on that equilibrium state. The real state of gothic vaults must be nearer, to that of the membrane state studied in (3) and (4).

Conclusions

The following conclusions can be drawn:

a) The actual condition for the stability of a masonry arch is that the "line of pression", or the excentricity of axial force, lies into the depth of the arch and not that the line of thrust lies into the depth of the arch.

b) Lines of pression and lines of thrust are the same only if they are straight lines without changes of slope.

c) Continuous load lines of pression and lines of thrust are always different, although very similar in most of cases.

d) For inclined loads, only the use of lines of pression is sure.
References


