Application of Bott-Duffin inverse to static and dynamic analysis of masonry structures

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Abstract

The present study deals with the Finite Element Analysis (FEA) of masonry structures by means of Bott-Duffin inverse. There are various kinds of structural problem about static and dynamic behaviors under the subsidiary condition of displacement such as contact problem, unilateral stress problem, homology design problem, etc. In masonry structures, due to the material properties, only compressive stress is assumed to exist and to a certain extent they become contact problem. Bott-Duffin inverse enables us to present an automatic analytical method for a system of simultaneous linear equations with the subsidiary condition of unknowns. In this paper, Bott-Duffin inverse is briefly introduced and its application to static and dynamic contact analyses of masonry structures are presented.

1 Introduction

The stability of many of historical masonry structures is now threatened by growing fractures, and how to repair and maintain for these structures becomes a weighty problem. The repair and maintenance of historical masonry structures require understanding of their structural behavior particularly up to collapse. A structural model of such masonry material is important for structural analysis by such as Finite Element Method (FEM). Definitely masonry material can resist high compressive stresses but only feeble tensions. Conventional assumptions on masonry are made such that no sliding
failure, no tensile strength and infinite compressive strength, and rigid behavior due to compression.

There are two approaches for the analysis of masonry structures by means of FEM, one is macro-modeling and the other is micro-modeling. The most widely used macro-modeling is based on the assumption of isotropy and homogeneity for material, Drucker-Prager plastic failure criterion with low-level cut-off on tensile stresses [1]. Other FEA non-linear models are based on the damage mechanics. Cracks are assumed to form in planes perpendicular to the direction of maximum principal tensile stress which reaches the specified tensile strength. The cracked masonry is anisotropic and smeared crack model is adopted [2]. Anisotropic continuum model [3, 4] and continuum model [5] are applied for masonry walls. For sufficiently large structures, the global response of masonry can be well predicted even without the inclusion of the local interaction between the masonry components. Distortion method [6] is also applied for masonry shells.

For the micro-modeling of masonry, composite interface model [3], mortar joint model [7], and elastic-plastic joint element [8] are applied for the non-linear behavior of masonry confining the elastic-plastic failure to mortar bed-joints. The micro-modeling is capable for describing the local interaction between masonry components, however, it becomes very difficult to solve for sizable masonry structures because of increment in the number of interfaces.

The first part of the present study covers a brief introduction of Bott-Duffin inverse [9-13]. The second part deals with an analytical method of masonry structures with inequality constraint conditions by means of the Bott-Duffin inverse.

2 Bott-Duffin inverse

2.1 Bott-Duffin inverse

Let us consider the simultaneous equation of unknown \( \mathbf{d} \), vector of order \( n \), and \( \mathbf{r} \), vector of order \( n \), with a subsidiary condition as

\[
\mathbf{Kd} + \mathbf{r} = \mathbf{f} \tag{1}
\]

\[
\mathbf{d} \in L \tag{2}
\]

\[
\mathbf{r} \in L^\perp \tag{3}
\]

\[
\mathbf{f} \in \mathbb{R}^n \tag{4}
\]

where \( \mathbf{K} \), symmetric positive definite matrix of order \( n \times n \), \( \mathbf{f} \), vector of order \( n \), \( \mathbb{R}^n \), linear space of order \( n \), \( L \), subspace in \( \mathbb{R}^n \), \( L^\perp \), orthogonal complement to \( L \). The subsidiary condition of eqns (2) and (3) is given by

\[
\mathbf{d}^T \mathbf{r} = 0 \tag{5}
\]
where $T$, symbol of transpose. If $a$ is any vector in $R^n$, $P_L$ and $P_{L\perp}$ are orthogonal projectors on $L$ and $L\perp$, eqns (1) and (5) take the form

$$d = P_La$$

$$r = P_{L\perp}a = f - KP_La$$

$$P_L + P_{L\perp} = I$$

where $I$, unit matrix of order $n \times n$. Substituting eqns (6) and (7) into eqn (1), we obtain

$$[KP_L + P_{L\perp}]a = f$$

If the coefficient matrix of order $n \times n$ of eqn (9) is nonsingular, eqns (1) and (5) are consistent for all $f$ and their solutions are unique. In this case, from eqn (9), we get

$$a = [KP_L + P_{L\perp}]^{-1}f$$

Substituting eqn (10) into eqns (6) and (7), and using eqn (1), we obtain

$$d = P_L[KP_L + P_{L\perp}]^{-1}f$$

$$r = f - Kd$$

The coefficient matrix of $f$ in the right side of eqn (11) is called “the Bott-Duffin inverse of $K$” and denoted by $K_{(L)}^{(-1)}$, which is orthogonal projector on $P_L$.

$$K_{(L)}^{(-1)} = P_L[KP_L + P_{L\perp}]^{-1}$$

The solution of eqn (1) becomes

$$d = K_{(L)}^{(-1)}f$$

$$r = K_{(L\perp)}^{(-1)}f$$

where $K_{(L\perp)}^{(-1)}$, is orthogonal projector on $P_{L\perp}$, which is called “the Bott-Duffin inverse of $K$” given by

$$K_{(L\perp)}^{(-1)} = P_{L\perp}[KP_L + P_{L\perp}]^{-1}$$

2.2 Basic equation type 1

Let us consider the simultaneous equation of unknown $d$ and $r$ with a subsidiary condition as

$$Kd + r = f$$

$$d^T r = 0$$
When eqns (17) and (18) are related to the problem of structural analysis, $K$, stiffness matrix in the incremental interval of order $n \times n$, $d$, incremental displacement vector of order $n$, $r$, incremental reaction vector of order $n$, $f$, incremental load vector of order $n$, respectively. Number of degrees of unknowns are $2n$, however, that of simultaneous equation are $n+1$. As the number of equations is not equal to that of unknowns, the solution(s) of the basic equation type 1 is not unique.

From eqns (14) and (15), the solutions are given by means of the Bott-Duffin inverse.

$$d = K^(-1) f = P_L[KP_L + P_{L\perp}]^{-1} f$$
$$r = K^(-1) f = P_{L\perp}[KP_L + P_{L\perp}]^{-1} f$$

2.3 Basic equation type 2

The solution(s) of the basic equation type 1 is not unique, therefore the basic equation type 2 is adopted here as

$$Kd + r = f$$
$$Ad = 0, \quad r = A^T \lambda$$

where $A$, the subsidiary matrix of order $m \times n$, $\lambda$, any vector of order $n$.

Let us prove the orthogonality condition of $d$ and $r$ by using eqn (22).

$$d^T r = d^T A^T \lambda = [Ad]^T \lambda = 0$$

Therefore, eqn (23) is equivalent to eqn (18).

The above basic equation type 2, eqns (21) and (22), are derived by minimizing the total potential energy function and the subsidiary condition.

$$\Pi = \frac{1}{2} d^T K d - f^T d$$
$$Ad = 0$$

Lagrange multiplier method can be applied to the analysis of the minimization problem of eqn (24) with the subsidiary condition of eqn (25). Introducing Lagrange multipliers $\lambda$, this problem becomes the minimization problem of unknowns $n+m$ without the subsidiary condition in which the independent variables are $d$ and $\lambda$. The total potential energy function becomes

$$\Pi_k = \frac{1}{2} d^T K d - f^T d + \lambda^T A d$$

The stationary conditions of eqn (26) are given by

$$\frac{\partial \Pi_k}{\partial d} = K d - f + A^T \lambda = 0$$
$$\frac{\partial \Pi_k}{\partial \lambda} = Ad = 0$$
If we introduce the notation

\[ r = A^T \lambda \]  \hspace{1cm} (29)

Then eqn (27) takes the form

\[ Kd + r = f \]  \hspace{1cm} (30)

The minimization problem with the subsidiary condition given by eqns (24) and (25) has resulted in the system of equations given by eqns (30) and (28), in other words, simultaneous equations with unknowns \( d \) and \( r \).

In the case of \( r = 0 \), displacement \( d \) is given by

\[ d = K^{-1} f \]  \hspace{1cm} (31)

This displacement, however, is not generally satisfied with eqn (28).

On the other hand, in the case of \( r \neq 0 \), the following equation is obtained by eqn (30)

\[ d = K^{-1} [f - r] \]  \hspace{1cm} (32)

Substituting eqn (32) into eqn (25), we obtain

\[ AK^{-1} [f - r] = 0 \]  \hspace{1cm} (33)

That is, \( r \) is virtual external load vector to be satisfied with the subsidiary condition (25). Eqns (21) and (22) are the special case of eqns (17) and (18), and their solutions are obtained uniquely because of the orthogonality condition (23).

2.4 Basic equation type 4

Let us consider the basic equation type 4 as

\[ Kd + r = f \]  \hspace{1cm} (34)
\[ Ad = g, \quad r = A^T \lambda \]  \hspace{1cm} (35)

in which the difference between type 2 and type 4 is the subsidiary condition of \( g \) in the right side of eqn (35).

Introducing the new variable \( z \), the vector \( d \) can be transformed as

\[ z = d - A^+ g \]  \hspace{1cm} (36)

where, \( A^+ \), the general inverse matrix of order \( n \times m \) to \( A \), which is given by

\[ A^+ = A^T [AA^T]^{-1} \]  \hspace{1cm} (37)
\[ AA^+ = I \]  \hspace{1cm} (38)
Substituting eqn (36) into eqn (35), we obtain

\[ A[z + A^+g] = g \]  
\[ Az + AA^+g = g \]

From the characteristic of the general inverse matrix of eqn (38), we get

\[ Az + Ig = g \]  
\[ Az = 0 \]

On the other hand, substituting eqn (36) into eqn (34), we obtain

\[ K[z + A^+g] + r = f \]  
\[ Kz + r = f - KA^+g \]

If we introduce the notation

\[ h = f - KA^+g \]

Then eqn (43) takes the form

\[ Kz + r = h \]

in which is the same as the basic equation type 2 given by eqn (21). From the above procedure, eqns (34) and (35) take the form

\[ Kz + r = h, \quad h = f - KA^+g \]  
\[ Az = 0, \quad r = A^T\lambda \]

**2.5 Equation of motion**

Let us consider the minimizing the instantaneous total potential energy function with a subsidiary condition as

\[ \Delta \Pi = \frac{1}{2} \Delta \ddot{d}^T M \Delta \dot{d} + \frac{1}{2} \Delta \dot{d}^T K(t) \Delta d - \Delta f^T \Delta d \]  
\[ A\Delta d = \Delta g \]

where, \( \Delta \ddot{d} \), incremental velocity vector of order \( n \), \( \Delta d \), incremental displacement vector of order \( n \), \( M \), mass matrix of order \( n \times n \), \( K(t) \), instantaneous stiffness matrix of order \( n \times n \), \( \Delta f \), incremental external load vector of order \( n \), \( A \), the subsidiary matrix of order \( m \times n \), \( \Delta g \), incremental vector of order \( m \), \( n \), number of degrees of freedom, \( m \), number of the subsidiary conditions (\( m < n \)), respectively. Eqn (49) is extended to the dynamic problem from eqn (24) and the subsidiary condition (50) is the same as the basic equation type 4 given by eqn (35).
Lagrange multiplier method is applied to the analysis of the minimization problem of eqn (49) with the subsidiary condition of eqn (50). Introducing Lagrange multipliers \( \Delta \lambda \), this problem becomes the minimization problem of unknowns \( n + m \) without the subsidiary condition in which the independent variables are \( \Delta d \) and \( \Delta \lambda \). The instantaneous total potential energy function becomes

\[
\Delta \prod_k = \frac{1}{2} \Delta d^T M \Delta d + \frac{1}{2} \Delta d^T K(t) \Delta d - \Delta f^T \Delta d + \Delta \lambda^T (A \Delta d - \Delta g)
\]

(51)

The stationary conditions of eqn (51) are given by

\[
M \Delta \ddot{d} + K(t) \Delta d - \Delta f + A^T \Delta \lambda = 0
\]

(52)

\[
A \Delta d = \Delta g
\]

(53)

in which \( K(t)^T = K(t) \) is used here. If we introduce the notation

\[
\Delta r = A^T \Delta \lambda
\]

(54)

Then eqn (52) takes the form

\[
M \Delta \ddot{d} + K(t) \Delta d + \Delta r = \Delta f
\]

(55)

In which eqn (54) is the subsidiary condition.

Taking into account damping, eqn (55) becomes eqn (56) by using incremental input ground motion \( \Delta d_0 \)

\[
M \Delta \ddot{d} + C(t) \Delta \dot{d} + K(t) \Delta d + \Delta r = -M \Delta \ddot{d}_0
\]

(56)

where, \( C(t) \), instantaneous damping matrix of order \( n \times n \).

The minimization problem with the subsidiary condition given by eqns (49) and (50) has resulted in the system of equations given by eqns (56) and (53), in other words, simultaneous equations with unknowns \( \Delta d \) and \( \Delta r \).

3 Numerical examples and discussion

The contact problems between voussoirs, a soil and a masonry structure are numerically analyzed in order to examine the validity of the procedure presented in the previous chapter. The first example is Mosca’s bridge with 45 m span, over the Dora Riparia, Turin, which is an application of the theorems by Castigliano to find out the true structure of masonry arch bridge subjected to dead load [14]. The radical thickness of the voussoirs increases from 1.5 m at the crown to 2.0 m at the springings. The Young's modulus, Poisson’s ratio and weight per unit volume using in our analysis are 20.58kN/mm², 0.0 and 26.5kN/m³, respectively. The portion above the masonry arch ring, only the weight is taken into consideration.
By means of the Bott-Duffin inverse, the numerical analysis for masonry arch begins with the subsidiary condition \( Ad = 0 \), that is contact state. Figure 1 shows the normal stress and the thrust line at the different stages of calculations. In the first stage, we can see the tensile stress in the meridional direction at the springing, that is \( r > 0 \). The tensile force cannot be transmitted between voussoirs, however, the condition \( r < 0 \) needs in masonry structures. The contact state changes into the free state if \( r < 0 \) becomes \( r = 0 \), and then the corresponding nodes will move freely. On the other hand, the shift from the free state to the contact state occurs if the corresponding nodal displacements become the same, and then compressive force can transmit between them \( (r < 0) \).

The thrust line, at the springing, to the limit of the middle-third of the effective compressed section. A section of the arch near the springings must be cracked. In the second stage of calculation, the thickness of the arch ring at the springing, reduces from 2.010 meter to 1.539 meter, are used. As shown in Figure 1(b), in the final stage of calculation, the arch ring has no tensile forces in the meridional direction within the masonry arch. The arch can be maintained by compressive forces transmitted everywhere within a mass of masonry. The second example is masonry arches subjected to centric and eccentric concentrated loads. The results are shown graphically in Figure 2.

![Figure 1: Normal stress and thrust line.](image1)

(a) first stage (b) final stage

![Figure 2: Deformation of masonry arch.](image2)

(a) centric load (b) eccentric load

A two-stories one-span masonry wall subjected to both vertical and horizontal loads is numerically analyzed. The dimensions of the wall and openings, the Young's modulus, Poisson's ratio and weight per unit volume are \( 3m \times 6m \times 0.25m, 1m \times 2m, 1m \times 1.5m, 20.58kN/mm^2, 0.0 \) and \( 22.6kN/m^3 \), respectively. As horizontal load increases, final failure is obtained by overturning of the wall (Figure 3).

A spring-mass model with two degrees of freedom is analyzed by means of the Bott-Duffin inverse \([15]\). Mass, stiffness, initial displacement and initial velocity are \( m_1 = m_2 = 10kN\cdot sec^2/cm, k_1 = 100kN/cm, k_2 = 300kN/cm, d_1 = d_2 = 10cm \) and \( d_1 = d_2 = 0cm/sec \), respectively. The shift from the contact state to the free state and its inversion can be seen in Figure 4(b).
4 Concluding remarks

Bott-Duffin inverse enables us to present an automatic analytical method for a system of simultaneous linear equations with the subsidiary condition of unknowns. The main advantage of the present method is that it allows the procedure without rebuilding the stiffness matrix $K$ even if the contact state changes. Numerical examples show the validity of the present method for masonry structures as a contact problem.

References


