Study with the Chebyshev series method of a strip on an elastic nonlinear Winkler-Pasternak-type foundation

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Abstract

This paper describes the procedure and results of theoretical investigations into the beam-soil foundation interaction. Actual pressure versus soil settlement response is represented by a nonlinear Winkler-Pasternak relation. The resulting nonlinear governing differential equation is solved using Chebyshev polynomials.

The iterative numerical procedure adopted to solve the resultant system of simultaneous equations is presented. The numerical results are plotted in the form of a chart and, for a range of values of non-dimensional parameters $\gamma$, $Z$ and $V$, and load $Q$, they are compared with the available F.D.M. results and are found to be in good agreement. Keywords: contact problems, Chebyshev polynomials, Winkler-Pasternak-type soil, nonlinear foundation, numerical methods.

1 Introduction

The theory of beams, plates and shells supported on elastic foundations has been the subject of numerous investigations.

The simplest approach depends on the assumption that the intensity of the foundation reaction at any point is proportional to the deflection of the structure at that point (Winkler-Pasternak model). As an extreme case, the foundation is represented as elastic half-space [1].

The first of the models provides a simpler solution to the problem of contact between structure and foundation, and for this reason it has been the one most adopted up to now. The problem, in its simplest formulation, appears as linear elastic. However, it does not have sufficient generality in the description of the
physical phenomenon of the results that are obtained as a consequence. In concrete applications, various sources of nonlinearity are important: for example, the possible regime of large deformations, the non-bilaterality of the contact constraint and the nonlinearity of the constitutive law of the materials (structure and/or foundation). Referring to the third cause of nonlinearity, it is known that where the foundation is represented by a foundation terrain, it is necessary to take into account a strongly nonlinear constitutive law [2].

The aforementioned contact problem, whether one adopts the elastic half-space model [3, 4] or the Winkler-Pasternak model [5, 6, 7, 8], can be solved with a significant saving of time and calculations compared to other numerical methods such as F.E.M. or F.D.M., by means of the calculation technique that relies on the Chebyshev series of orthogonal polynomials which, compared to other polynomials currently in use in engineering applications, provide a rapid convergence towards the solution.

In this paper we consider foundation beams lying on a soil foundation for which we adopt a Winkler-Pasternak model of behaviour made nonlinear. To take into account the nonlinearity of the pressure-settlement law \( q(w) \), we shall include a third-degree term having as its coefficient nonlinearity constant \( N \), which is added to parameters \( K \) and \( G \) of the classic linear model.

We shall illustrate in detail the application of the Chebyshev series in solving the problem.

Examples of applications and comparisons of results with those deriving from the application of the traditional method of finite differences will allow evaluation of the validity and accuracy of the method.

2 Problem formulation

The relation between the pressure and the deflection of the foundation surface for the nonlinear Winkler-Pasternak-type soil foundation is described by:

\[
q = K \cdot w - G \cdot (\nabla^2 w) - N \cdot w^3
\]  

(1)

It is assumed that the beam or plate deforms by bending only. The differential relation between acting load and deflection is given by:

\[
(\nabla^2)^2 w + \frac{1}{D}(K \cdot w - G \cdot (\nabla^2 w) - N \cdot w^3) = \frac{1}{D} q
\]  

(2)

where we have the following symbols:

- \( \nabla^2 \) Laplace’s operator;
- \( w \) vertical displacement;
- \( D \) flexural stiffness;
- \( K \) Winkler constant;
- \( G \) Pasternak constant;
- \( N \) nonlinear parameter of foundation material;
- \( q \) uniformly distributed load.

Using various dimensionless parameters such as:
\[ \eta = \frac{w}{L} \; ; \; Z = \left( \frac{K}{D} \right) \cdot L^4 \; ; \; V = \left( \frac{G}{D} \right) \cdot L^2 \; ; \; \gamma = \left( \frac{N}{D} \right) \cdot L^6 \; ; \; Q = \left( \frac{q}{D} \right) \cdot L^3 \]  

(3)

eqn (2) takes the form:

\[ (\nabla^2)^2 \eta + Z \cdot \eta - V \cdot (\nabla^2 \eta) = Q + \gamma \cdot \eta^3 \]  

(4)

Using \( E \) to indicate the elastic longitudinal module and \( \nu \) to indicate the transversal contraction coefficient of the material, and appropriately specifying the Laplace operator, we can particularize eqn (4) to the study of plates (rectangular or circular) or beams.

In the case of a beam with a rectilinear axis of width \( b \) and height \( h \) we have \( D = E \cdot b \cdot h^3/12 \). Using \( x = \chi/L \) to indicate the current abscissa made dimensionless with respect to half-span \( L \) of the beam we have:

\[ \nabla^2 = \left( \frac{d^2}{dx^2} \right) \]  

(5)

The specification of the boundary conditions depending on external constraints, to associate with eqn (4), allows completion of the mathematical formulation of the problem at hand.

![Figure 1: “Chebyshev points” for the first six polynomials.](image)

3 Chebyshev polynomials

In the normalization interval \([-1, 1]\), in which the weight function is defined by:

\[ g(x) = \sqrt{1-x^2} \]  

(6)

the Chebyshev polynomials of the first kind (Scheid [9]) are given by:

\[ T_n(x) = \cos(n \cdot \arccos x), \quad n = 0, 1, 2, \ldots \]  

(7)

and satisfy the relation of orthogonality:
\[
\int_{-1}^{+1} \frac{T_m(x) \cdot T_n(x)}{\sqrt{1-x^2}} \, dx = \begin{cases} 
0 & \text{if } m \neq n \\
\frac{\pi}{2} & \text{if } m = n \neq 0 \\
\frac{\pi}{2} & \text{if } m = n = 0 
\end{cases}
\]  

(8)

If the integration interval is normalized to \([0, 1]\), it can be shown that the orthogonal family \(T^*(x)\) of Chebyshev polynomials can be found from the following three-term recurrence relation:

\[
T^*_0(x) = 1 \\
T^*_1(x) = 2x - 1 \\
T^*_n(x) = 2(2x - 1) \cdot T^*_{n-1}(x) - T^*_{n-2}(x) \quad \text{with } n \geq 2
\]

(9)

The polynomial \(T^*_n(x)\) possesses \(n+1\) extremal points in \([0, 1]\) and its first derivative vanishes at each of the extremal points in the interior of the interval. The set of points at which \(T^*_n(x)\) attains its norm with alternating signs are known as “Chebyshev points”.

4 The Chebyshev method of solution for a beam

To solve the problem of a beam on a nonlinear Winkler-Pasternak-type soil foundation, we assume that the deflexion of the surface of the beam can be expressed by mean of the series of powers:

\[
\eta = \sum_{i=0}^{n} C_i \cdot \rho^i
\]

(10)
in which the \(C_i\) identify \((n+1)\) unknown coefficients, to be determined by means of eqn (4) and the end conditions.

\[
\begin{array}{c}
1 \\
1 \\
Q \\
\eta
\end{array}
\]

Figure 2: The simply supported beam considered in this study.

Substituting eqn (10) in eqn (4) and equating coefficients of like powers of \(x\) on both sides of eqn (4), leads to \((n+16)\) simultaneous equations. In addition, end conditions yield four equations. Therefore, this makes \((n+20)\) simultaneous equations to determine only \((n+1)\) unknowns \((C_0, C_1, \ldots, C_n)\).

To obtain a univocal solution for the \(C_i\), the previous equation is “perturbed” by adding the following quantity to the second member:
thus obtaining:

\[
\sum_{i=4}^{n} i(i-1)(i-2)(i-3) \cdot C_i \cdot x^{i-1} + Z \cdot \sum_{i=0}^{n} C_i \cdot x^{i-3} - V \cdot \sum_{i=2}^{n} i(i-1) \cdot C_i \cdot x^{i+1} = Q \cdot x^3 + \gamma \cdot \left( \sum_{i=0}^{n} C_i \cdot \rho^{i+1} \right)^3 + \sum_{i=n+1}^{n+19} Ci \cdot T^*(i-4)(x)
\]

(12)

where \(T^*_n(x)\) stands for the Chebyshev polynomial of degree \(n\) evaluated in the normalized integration interval \([0, 1]\).

As concerns symmetry conditions, it is found that at the centre of the beam (for \(x = 0\)) there must be:

\[
\frac{d\eta}{dx} = 0, \quad \frac{d^3\eta}{dx^3} = 0 \quad (13)
\]

The end conditions (for \(x = 1\)) also supply for the simply supported beam:

\[
\eta = 0, \quad \frac{d^2\eta}{dx^2} = 0 \quad (14)
\]

Taking into account eqn (10) assigned to deflection \(\eta\) of the beam, the preceding eqn (13) and eqn (14) results respectively:

\[
C_1 = 0, \quad C_3 = 0 \quad (13\text{bis})
\]

(14bis)

As anticipated, by equating the coefficients of equal power of \(x\) in the two members of eqn (12), we arrive at the writing of twenty-two algebraic equations. Associating with them the two eqs (13bis) and the two equations corresponding to the specific end conditions eqs (14bis), we obtain a system composed of twenty-six nonlinear equations.

For the solution of this system which, owing to nonlinearity, must be obtained iteratively by updating the vector of the known terms, we made use of the \textit{linbcg} routine present in the Numerical Recipes [10]. The mathematical procedure is given in the Appendix.

5 Results of numerical processing and conclusions

Let us consider the issue of the beam subjected to a uniformly distributed load in the case of the simply supported ends.

The series of powers in eqn (10) adopted to describe the deflection of the surface of the beam is illustrated in detail by assuming \(n = 6\).

The situation of the soil of reference was that of Winkler-Pasternak soils with \(Z=75\) and \(V=2\) and of Winkler type with \(Z=75\) and \(V=0\).

Nonlinearity conditions were characterized by different values of dimensionless parameter \(\gamma\). The beam was studied with reference to values of \(\gamma\) amounting respectively to 0.0 (linear case), 10 000, 30 000 and 50 000.
Finally, the numerical calculations were performed with the adoption of the values 1.00 and 0.75 for nondimensional parameter $Q$ of the external load, which was considered in every circumstance as being uniformly distributed over the beam.

As a result of the numerical calculations performed, the coefficients $C_i$ of eqn (10), which characterize the deflection of the beam, were obtained. In the case of load $Q=1$ and Winkler-Pasternak-type soil, the deflection was represented graphically in Figure 3.

![Figure 3: Displacements of the beam with load $Q=1$ and $V=2$.](image)

In some of the cases studied, maximum displacement was compared to the analogous result obtained with the finite differences method.

<table>
<thead>
<tr>
<th>$Q$</th>
<th>$\gamma$</th>
<th>$V$</th>
<th>Chebyshev</th>
<th>F.D.M.</th>
<th>$E$ %</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>0.0</td>
<td>0</td>
<td>0.0149055</td>
<td>0.0149943</td>
<td>0.59</td>
</tr>
<tr>
<td>1</td>
<td>10 000</td>
<td>0</td>
<td>0.0153540</td>
<td>0.0153739</td>
<td>0.13</td>
</tr>
<tr>
<td>1</td>
<td>30 000</td>
<td>0</td>
<td>0.016507</td>
<td>0.0163535</td>
<td>-0.93</td>
</tr>
<tr>
<td>1</td>
<td>50 000</td>
<td>0</td>
<td>0.017807</td>
<td>0.0179560</td>
<td>0.82</td>
</tr>
<tr>
<td>0.75</td>
<td>0.0</td>
<td>0</td>
<td>0.0111791</td>
<td>0.0112452</td>
<td>0.59</td>
</tr>
<tr>
<td>0.75</td>
<td>10 000</td>
<td>0</td>
<td>0.0113580</td>
<td>0.0114008</td>
<td>0.38</td>
</tr>
<tr>
<td>0.75</td>
<td>30 000</td>
<td>0</td>
<td>0.0117772</td>
<td>0.0117533</td>
<td>-0.20</td>
</tr>
<tr>
<td>0.75</td>
<td>50 000</td>
<td>0</td>
<td>0.0122900</td>
<td>0.0121833</td>
<td>-0.88</td>
</tr>
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</table>
Table 2: Percentage differences between the maximum displacements of the beam on the Winkler-Pasternak-type soil (V=2).

<table>
<thead>
<tr>
<th>$Q$</th>
<th>$\gamma$</th>
<th>$V$</th>
<th>Chebyshev</th>
<th>F.D.M.</th>
<th>E %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>2</td>
<td>0.0144211</td>
<td>0.0141477</td>
<td>-1.93</td>
</tr>
<tr>
<td>1</td>
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<td>0.0147870</td>
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<td>-2.37</td>
</tr>
<tr>
<td>1</td>
<td>30 000</td>
<td>2</td>
<td>0.0157040</td>
<td>0.0151706</td>
<td>-3.52</td>
</tr>
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<td>50 000</td>
<td>2</td>
<td>0.0170770</td>
<td>0.0163090</td>
<td>-4.70</td>
</tr>
<tr>
<td>0.75</td>
<td>0.0</td>
<td>2</td>
<td>0.0108158</td>
<td>0.0106108</td>
<td>-1.93</td>
</tr>
<tr>
<td>0.75</td>
<td>10 000</td>
<td>2</td>
<td>0.0109650</td>
<td>0.0107325</td>
<td>-2.17</td>
</tr>
<tr>
<td>0.75</td>
<td>30 000</td>
<td>2</td>
<td>0.0112990</td>
<td>0.0110028</td>
<td>-2.69</td>
</tr>
<tr>
<td>0.75</td>
<td>50 000</td>
<td>2</td>
<td>0.0117050</td>
<td>0.0113195</td>
<td>-3.40</td>
</tr>
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</table>

Tables 1 and 2, which correspond respectively to the Winkler-type and Winkler-Pasternak-type soil foundation, give the percentage differences of displacements at the centre of the beam evaluated with the two methods for each situation analysed.

The resulting percentage differences are in all cases below 5%. Convergence in all cases was quite rapid and the numerical calculations performed with the Chebyshev series method in all cases required a number of iterations smaller than what was necessary with the finite differences method.

These conclusions demonstrate that the method ensures accuracy in results and provides applicative advantages with respect to traditional numerical procedures.

References


As anticipated, we obtain a system composed of twenty-six nonlinear equations and, of these, twelve are solved immediately by determining in cascade the unknowns from $C_{14}$ to $C_{25}$.

\[
C_{25}=1.0/(2199023255552 \cdot (-\gamma \cdot B_{21}))
\]
\[
C_{24}=1.0/(549755813888 \cdot (-\gamma \cdot B_{20}+23089744183296 \cdot C_{25}))
\]
\[
C_{23}=1.0/(137438953472 \cdot (-\gamma \cdot B_{19}-12562502893568 \cdot C_{25}+5497558138880 \cdot C_{24}))
\]
\[
C_{22}=1.0/(34359738368 \cdot (-\gamma \cdot B_{18}+338168545017856 \cdot C_{25}-25426206392320 \cdot C_{24}+1305670057984 \cdot C_{23}))
\]
\[
C_{21}=1.0/(8589934592 \cdot (-\gamma \cdot B_{17}-700809813688 \cdot C_{25}+72155450572800 \cdot C_{24}-5712306503680 \cdot C_{23}+309237645312 \cdot C_{22}))
\]
\[
C_{20}=1.0/(2147483648 \cdot (-\gamma \cdot B_{16}+1062579203997696 \cdot C_{25}-40552804761600 \cdot C_{24}+15260018802688 \cdot C_{23}-73014444032 \cdot C_{22}))
\]
\[
C_{19}=1.0/(536870912 \cdot (-\gamma \cdot B_{15}-121999834533068 \cdot C_{25}+1991834033192666 \cdot C_{24}-27827093110784 \cdot C_{23}+3195455668224 \cdot C_{22}-282930970624 \cdot C_{21}+17197869184 \cdot C_{20}))
\]
\[
C_{18}=1.0/(134217728 \cdot (-\gamma \cdot B_{14}+1083059755548672 \cdot C_{25}-12364657950720 \cdot C_{24}+36681168191488 \cdot C_{23}-5429778186240 \cdot C_{22}+661693399040 \cdot C_{21}-62277025792 \cdot C_{20}+4026531840 \cdot C_{19}))
\]
\[
C_{17}=1.0/(33554432 \cdot (-\gamma \cdot B_{13}-7525672256612864 \cdot C_{25}+173752901959600 \cdot C_{24}-36108024938496 \cdot C_{23}+6620826304512 \cdot C_{22}-1042167103488 \cdot C_{21}+135291469824 \cdot C_{20}-13589544960 \cdot C_{19}+939524096 \cdot C_{18}))
\]
\[
C_{16}=1.0/(8388608 \cdot (-\gamma \cdot B_{12}+411758179123200 \cdot C_{25}-110292369408000 \cdot C_{24}+27039419596800 \cdot C_{23}-5977134858240 \cdot C_{22}+1167945891840 \cdot C_{21}-196293427200 \cdot C_{20}+27262976000 \cdot C_{19}-2936012800 \cdot C_{18}+218103808 \cdot C_{17}))
\]
\[
C_{15}=1.0/(2097152 \cdot (-\gamma \cdot B_{11}-177570714746880 \cdot C_{25}+54553214976000 \cdot C_{24}-15547666268160 \cdot C_{23}+4063273943040 \cdot C_{22}-959384125440 \cdot C_{21}+200655503360 \cdot C_{20}-36175872000 \cdot C_{19}+5402263552 \cdot C_{18}+627048448 \cdot C_{17}+50331648 \cdot C_{16}))
\]
\[
C_{14}=1.0/(5242880 \cdot (-\gamma \cdot B_{10}+60144919511040 \cdot C_{25}-21002987765760 \cdot C_{24}+6880289095680 \cdot C_{23}-2095125626880 \cdot C_{22}+586290298880 \cdot C_{21})
\]

Appendix
The nonlinear algebraic system that solves the problem of the beam is defined by the remaining fourteen equations and is written symbolically:

\[ [M] - \{ C \} = \{ q \} \quad (15) \]

where \( \{ C \} \) indicates the vector of the unknowns \( C_0, \ldots, C_{13} \) and \([M]\) the relative matrix of the coefficients.

The solution is obtained iteratively owing to the nonlinearity of the vector of the known terms \( \{ q \} \). This vector is obtained, at the generic pass \( j \), by initially calculating the terms \( B_i \) (with \( i = 3, \ldots, 21 \)) and as a function of the values \( C_k \) (with \( k = 0, \ldots, 6 \)) calculated at the previous pass \( j - 1 \). Knowing the terms \( B_i \) (with \( i = 3, \ldots, 9 \)) we calculate in cascade the terms \( C_r \) (with \( r = 14, \ldots, 25 \)) and finally vector \( \{ q \} \) as a function of the uniformly distributed load expressed in nondimensional terms \( Q \), of the nonlinear, dimensionless coefficient \( \gamma \), of the terms \( B_i \) (with \( i = 3, \ldots, 9 \)) and of the \( C_r \).

We give below, in order, terms \( B_i \) (\( i = 3, \ldots, 21 \)), the components of vector \( \{ q \} \) and matrix \([M]\) of the coefficients.

Coefficients \( B_i (i = 3, \ldots, 21) \) of the series in eqn (10) are obtained as a function of terms \( C_0 \ldots C_6 \):
Finally, matrix $[M]$ of the coefficients of the solving system is given by:
Table 3.

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