Axisymmetric indentation of an elastic layer on a rigid foundation with a circular hole

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Abstract

The axisymmetric indentation problem of an elastic layer overlaid on a rigid foundation with a circular hole is investigated. An infinite rigid punch is pressed onto the upper surface of the elastic layer causing a small deformation mode. The problem is equivalent to a mixed boundary-value problem of the three-dimensional theory of elasticity, and an analytical solution is obtained through an infinite system of simultaneous equations by expressing the normal displacement in the circular hole as an appropriate series. Significant effects of the layer thickness and circular hole on the stress fields are demonstrated with numerical results.

Keywords: elasticity, contact problem, indentation, rigid punch, elastic layer, theoretical analysis.

1 Introduction

Contact problems have attracted considerable attention due to possible application to various important problems and have constituted one of the fields in the theory of elasticity since the work of Hertz [1]. Hertz theory of contact has been expanded by Boussinesq [2] and has been developed to important applications by numerous authors. These problems are equivalent to mixed boundary-value problems where stresses and displacements induce in an elastic half-space. However, the mixed boundary-value problem of an elastic layer has been dealt with in a relatively few publications. The axisymmetric indentation problems of an elastic layer resting without friction on a rigid foundation, or attached to a rigid foundation by a frictionless rigid punch, have been considered...
by Lebedev and Ufliand [3], Hayes et al. [4], Sakamoto et al. [5], Haider and Holmes [6], Wang and Lakes [7] and Yang [8].

The indentation problems of a layer resting on a rigid foundation involving a circular hole were considered by Low [9], Dhaliwal and Singh [10] and Woźniak et al. [11]. In each case, the problem was reduced to the solution of Fredholm integral equations.

In this study, an analytical solution for the contact between an infinite rigid punch and an infinite elastic layer resting on a rigid foundation with a circular hole is presented. Instead of using Fredholm integral equations, the solution is obtained through an infinite system of simultaneous equations by expressing the normal displacement of the layer in the circular hole as an appropriate series. Convergence can be achieved using less than 8 terms of the series. Numerical results are obtained clarifying the effects of layer thickness and circular hole on the stress fields.

2 Formulation of the problem and its solution

A cylindrical coordinate system \((r, \theta, z)\) is used in this study. We consider an infinite, isotropic, elastic layer with thickness \(h\), as shown in Fig. 1, which is indented by an infinite rigid punch. The layer resting on a rigid foundation weakened by a circular hole with radius \(a\). If surfaces of the layer are frictionless, the boundary conditions of the layer can be described by the following equations:

\[
\begin{align*}
(w_z)_{z=0} & = 0, \quad (a \leq r < \infty), \quad (1a) \\
(\sigma_z)_{z=0} & = 0, \quad (0 \leq r < a), \quad (1b) \\
(\tau_{rz})_{z=0} & = 0, \quad (0 \leq r < \infty), \quad (1c) \\
(w_z)_{z=h} & = -\delta, \quad (0 \leq r < \infty), \quad (1d) \\
(\tau_{rz})_{z=h} & = 0, \quad (0 \leq r < \infty), \quad (1e)
\end{align*}
\]

![Figure 1: Elastic layer indented by an infinite rigid punch.](image)
where $\delta$ is the indented displacement. The solution of elastic equilibrium equations without torsion can be derived using Boussinesq’s harmonic stress functions $\varphi_0$ and $\varphi_3$, i.e.

\[
2G_{u_r} = \partial \varphi_0 / \partial r + z \partial \varphi_3 / \partial r, \quad \nu_\theta = 0,
\]
\[
2G_{w_z} = \partial \varphi_0 / \partial z + z \partial \varphi_3 / \partial z - (3 - 4\nu)\varphi_3,
\]
\[
\sigma_r = \partial^2 \varphi_0 / \partial r^2 + z \partial^2 \varphi_3 / \partial r^2 - 2\nu \partial \varphi_3 / \partial z,
\]
\[
\sigma_\theta = \partial \varphi_0 / (r \partial r) + z \partial \varphi_3 / (r \partial r) - 2\nu \partial \varphi_3 / \partial z,
\]
\[
\sigma_z = \partial^2 \varphi_0 / \partial z^2 + z \partial^2 \varphi_3 / \partial z^2 - 2(1 - \nu)\partial \varphi_3 / \partial z,
\]
\[
\tau_{rz} = \partial^2 \varphi_0 / \partial r \partial z + z \partial^2 \varphi_3 / \partial r \partial z - (1 - 2\nu)\partial \varphi_3 / \partial r,
\]
\[
\tau_{r\theta} = \tau_{\theta e} = 0,
\]  

(2)

where $G$ is the shear modulus and $\nu$ is Poisson’s ratio of the elastic solid, respectively. Functions $\varphi_0$ and $\varphi_3$ satisfy the following equation:

\[
\nabla^2 \varphi_0 = \nabla^2 \varphi_3 = 0,
\]
\[
\nabla^2 \equiv \partial^2 / \partial r^2 + \partial / (r \partial r) + \partial^2 / \partial z^2.
\]  

(3)

If the stress functions $\varphi_0$ and $\varphi_3$ are chosen so as to satisfy Eq. (3), these two functions could be written in the following equations:

\[
\varphi_0 = -\nu G \delta (2z^2 - r^2) / h + \int_0^\infty \{A(\lambda) \cosh \lambda z + B(\lambda) \sinh \lambda z\} J_0(\lambda r) d\lambda,
\]
\[
\varphi_3 = G \delta z / h + \int_0^\infty \{C(\lambda) \sinh \lambda z + D(\lambda) \cosh \lambda z\} J_0(\lambda r) d\lambda,
\]  

(4)

where $J_0(\lambda r)$ is the Bessel function of the first kind of order $n = 0$ and $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ are functions which can be obtained by matching appropriate boundary conditions. By substituting Eq. (4) into Eq. (2) and using boundary conditions (1c), (1d) and (1e), we obtain

\[
\lambda A(\lambda) = \{\lambda h - (1 - 2\nu) \sinh \lambda h \cdot \cosh \lambda h\} D(\lambda) / \sinh^2 \lambda h,
\]
\[
\lambda B(\lambda) = (1 - 2\nu) D(\lambda),
\]
\[
C(\lambda) = -\coth \lambda h D(\lambda).
\]  

(5)

The normal displacement $w_z$ and stress $\sigma_z$ of Eq. (2) can be derived using functions $\varphi_0$ and $\varphi_3$ of Eq. (4). The unknown function $D(\lambda)$ can be determined by substitute $w_z$ and $\sigma_z$ into boundary conditions (1a) and (1b), i.e.
\[ (w_z)_{z=0} = -\frac{1-v}{G} \int_0^\infty D(\lambda)J_0(\lambda r) d\lambda = 0, \quad (a \leq r < \infty), \quad (6) \]
\[ (\sigma_z)_{z=0} = -\frac{2(1+v)G\delta}{h} + \int_0^\infty \lambda s(\lambda)D(\lambda)J_0(\lambda r) d\lambda = 0, \quad (0 \leq r < a), \quad (7) \]

where
\[ s(\lambda) = (\lambda h + \sinh \lambda h \cdot \cosh \lambda h) / \sinh^2 \lambda h. \quad (8) \]

The dual integral equations (6) and (7) are usually transformed into a Fredholm integral equation. In the present study, a different technique is utilized, where the normal displacement \((w_z)_{z=0}\) in the circular hole is expressed as an appropriate series function \([12]\), i.e.

\[ (w_z)_{z=0} = -\frac{4}{\pi ar} \sum_{n=0}^\infty \alpha_n U_{2n+2}(r/a), \quad (0 \leq r \leq a), \quad (9) \]

where \(\alpha_n\) \((n=0, 1, 2, \ldots)\) are unknown coefficients and \(U_{2n+2}(r/a)\) represents Tchebycheff function of the second kind. If we write
\[ Z_n(\lambda) = J_{n+1/2}(\lambda a / 2) \cdot J_{n-(1/2)}(\lambda a / 2), \quad (n = 0, 1, 2, \ldots), \]
\[ F_n(\lambda) = \lambda[Z_n(\lambda) - Z_{n+1}(\lambda)], \quad (n = 0, 1, 2, \ldots), \quad (10) \]

the following integral formula for Beesel function can be obtained:
\[ \int_0^\infty J_0(\lambda r)F_n(\lambda) d\lambda = \begin{cases} 4U_{2n+2}(r/a) / \pi ar, & (0 \leq r \leq a), \\ 0, & (a \leq r < \infty). \end{cases} \quad (11) \]

Using Eq. (11), we fined that the Hankel inversion of Eqs. (6) and (9) yields
\[ D(\lambda) = \frac{G}{1-v} \sum_{n=0}^\infty \alpha_n F_n(\lambda). \quad (12) \]

Substituting Eq. (12) into Eq. (7) and using following Gegenbauer’s formula:
\[ J_0(\lambda r) = \sum_{m=0}^{\infty} (2 - \delta_{0m}) X_m(\lambda) \cos m\phi, \quad (r = a \sin(\phi / 2)), \]
\[ X_m(\lambda) = J_m^2(\lambda a / 2), \quad (m = 0, 1, 2, \ldots), \quad (13) \]
we obtain
\[ \sum_{n=0}^{\infty} a_n \int_{0}^{\infty} \lambda s(\lambda) F_n(\lambda) \sum_{m=0}^{\infty} (2 - \delta_{0m}) X_m(\lambda) \cos m\phi d\lambda = 1, \quad (0 \leq r < a), \quad (14) \]

where \( \delta_{0m} \) is Kronecker’s delta function and
\[ a_n = h \alpha_n / \{2(1 + \nu)(1 - \nu)\delta\}, \quad (n = 0, 1, 2, \ldots). \quad (15) \]

Matching the coefficients of \( \cos m\phi \) on both side of Eq. (14), and taking the different between the \( m \)-th and \( (m+2) \)-th equations in Eq. (14), we obtain the following infinite system of simultaneous equations for determined the coefficients \( a_n \):
\[ \sum_{n=0}^{\infty} a_n \int_{0}^{\infty} s(\lambda) Y_m(\lambda) F_n(\lambda) d\lambda = \delta_{0m}, \quad (m = 0, 1, 2, \ldots), \quad (16) \]

where
\[ Y_m(\lambda) = \lambda^2 X_m(\lambda) - X_{m+2}(\lambda), \quad (m = 0, 1, 2, \ldots). \quad (17) \]

From Eqs. (12) and (15), \( D(\lambda) \) can be rewritten as
\[ D(\lambda) = E \delta \sum_{n=0}^{\infty} a_n F_n, \quad (18) \]

where \( E \) denotes Young’s modulus. The normal displacement \( (w_z)_{z=0} \) in the hole in Eq. (9) can be rewritten as
\[ \frac{h}{(1 + \nu)(1 - \nu)\delta} (w_z)_{z=0} = -\frac{8}{\pi a r} \sum_{n=0}^{\infty} a_n U_{2n+2} (r / a), \quad (0 \leq r \leq a). \quad (19) \]

If the value of \( \lambda \) is large, \( s(\lambda) \) and \( \lambda F_n(\lambda) \) can be asymptotically expressed as follows:
\[ s(\lambda) \rightarrow 1, \quad (20) \]
\[ \lambda F_n(\lambda) \rightarrow -\frac{8(n + 1)}{\pi a^2} \cos \lambda a. \quad (21) \]

Then, the normal contact stress \( (\sigma_z)_{z=0} \) between the elastic layer and the rigid foundation can be expressed as
\[
\frac{h}{E\delta}(\sigma_z)_{z=h} = -1 + \sum_{n=0}^{\infty} a_n \int_{0}^{\infty} \{s(\lambda) - 1\} \lambda F_n(\lambda) J_0(\lambda r) d\lambda \\
+ \sum_{n=0}^{\infty} a_n \int_{0}^{\infty} \left\{ \lambda F_n(\lambda) + \frac{8(n+1)}{\pi a^2} \cos \lambda a \right\} J_0(\lambda r) d\lambda \\
- \frac{8}{\pi a^2 \sqrt{r^2 - a^2}} \sum_{n=0}^{\infty} (n+1)a_n, \quad (a < r < \infty).
\]

3 Numerical results and discussion

To determine the coefficients \(a_n\) discussed in the previous section, it is necessary to evaluate the infinite integral of Eq. (16). If a large constant value \(\lambda_0\) is chosen, the infinite integral can be rewritten in the following form:

\[
\int_{0}^{\infty} s(\lambda) Y_m(\lambda) F_n(\lambda) d\lambda = \int_{0}^{\infty} \{s(\lambda) - 1\} Y_m(\lambda) F_n(\lambda) d\lambda \\
+ \int_{0}^{\lambda_0} Y_m(\lambda) F_n(\lambda) d\lambda + \int_{\lambda_0}^{\infty} Y_m(\lambda) F_n(\lambda) d\lambda.
\]

When \(\lambda\) is large, the asymptotic value of \(Y_m(\lambda) F_n(\lambda)\) can be derived as follows:

\[
Y_m(\lambda) F_n(\lambda) \rightarrow \frac{128(-1)^m (m+1)(n+1) \cos^2 \lambda a}{\pi^2 a^4 \lambda^2}.
\]

By substituting Eq. (24) into the last integral of Eq. (23), we obtain

\[
\int_{\lambda_0}^{\infty} Y_m(\lambda) F_n(\lambda) d\lambda = \frac{128(-1)^m (m+1)(n+1)}{\pi^2 a^3 \lambda_0^3} \left\{ \frac{\cos^2 \lambda_0 a}{\lambda_0^2} + \sin(2\lambda_0 a) \right\},
\]

where \(\sin(x)\) is the integral sine function. The first and second integrals of Eq. (23) can be calculated by the Simpson’s rule and \(\lambda_0 = 1500\) is used.

The radial distributions of normalized normal displacement in the circular hole \((\overline{w}_z)_{z=0} = h(w_z)_{z=0} / a\delta(1+\nu)(1-\nu)\) and normalized normal stress on the rigid foundation \((\overline{\sigma}_z)_{z=0} = h(\sigma_z)_{z=0} / E\delta\) are shown in Figs. 2 and 3, respectively. It is known that the dimensionless displacement \((\overline{w}_z)_{z=0}\) and stress \((\overline{\sigma}_z)_{z=0}\) are independent from the material constants, i.e., \(E\) and \(\nu\).

To investigate the effect of the layer thickness on the normal displacement and stress, Figs. 2 and 3 show the distribution of \((\overline{w}_z)_{z=0}\) and \((\overline{\sigma}_z)_{z=0}\) with variations of \(h / a\), respectively. It is interesting to note that the slope of \((\overline{w}_z)_{z=0}\) approaches infinity as \(r \to a_0\). It is shown that the magnitude of \((\overline{w}_z)_{z=0}\) increases with \(h / a\). The normalized contact stress \((\overline{\sigma}_z)_{z=0}\) has singularity at the edge of the hole. It is found that \((\overline{\sigma}_z)_{z=0}\) in the contact region is always compressive and the magnitude of \((\overline{\sigma}_z)_{z=0}\) increases with \(h / a\).
Figure 2: Radial distribution of normalized normal displacement.

Figure 3: Radial distribution of normalized normal contact stress.
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References


