ABSTRACT

The present paper deals with the dynamic analysis of linear single degree of freedom (SDOF) systems with random mass, random stiffness and random damping subject to random seismic loading. The importance of considering random system model is implemented with a detailed examination of the impulse response function of SDOF systems and with several exact solutions. Response characteristics are initially determined on condition of a specific realization of the structural system. Then, total probability theorem is applied to evaluate unconditional expectations of the response. Conventional zeroth, first and second order perturbation solutions are compared to the exact solutions for the response statistics. Random damping and random stiffness effects are investigated separately considering eigenvibrations, stationary response under harmonic loadings and horizontal nonstationary earthquake excitations. For eigenvibrations, in both cases, perturbation solutions carry divergent secular terms. When only random stiffness is considered, due to the secular divergent terms, perturbation solutions blow up with time, even for very small variabilities of the random variables. In the case of only random damping, as the divergent secular terms are under the governing control of the exponential decay, the existing deviations in the perturbed solutions become neither observable nor important. Hence, perturbation solutions don’t diverge from the exact solution and are observed to be very good approximations. Under harmonic loadings, after the dissipation of eigen vibrations, perturbation solutions don’t possess any secular terms and are very good approximations except for the cases in which the system is excited at or close to its resonance frequency. Finally, the nonstationary random response of stochastic SDOF systems subject to nonstationary random seismic exci-
tations are studied. It is concluded that perturbation solutions are good approximations in the case of random damping. However, in the case of random stiffness, significant deviations are observed, especially for short duration earthquakes. On the other hand, perturbations solutions seem to be admissible and useful in the estimation of the maximum variance of the response which is expected within the first few natural periods.

INTRODUCTION

Random vibrations theory of deterministic structures, in which only the loading is considered as a random process, has been well developed. Several papers, review articles and books have been published in this field. The present study applies the principles of random vibrations theory of deterministic structures to random vibrations theory of stochastic structures. Only SDOF systems will be covered in this paper. Based on the fundamentals of this study, the extension to multi-degree of freedom systems will be investigated in a forthcoming paper.

All structures possess uncertainties in their material properties and/or geometry which are due to physical imperfections, model inaccuracies and system complexities. Such uncertainties are spatially distributed over the structure and can be mathematically modeled using either random variables or random processes which may be functions of time and/or space. Random vibration analysis of stochastic structures considers random structural models under random loading. It should be pointed out that the utilized random model must fulfill all the physical requirements of the problem with probability 1.

Stochastic structural models have been considered by several researchers. In this respect, eigenvalue analysis of the random systems was attacked about twenty-five years ago and some reconsiderations or applications of earlier studies appeared recently. Random eigenvalue problems have been discussed by Boyce [5], Fox and Kapoor [24], Collins and Thompson [9], Shinozuka and Astill [40], Hasselman and Hart [29], Grigoriu [27], Nagashima and Tstusumi [38], and, Zhu and Wu [55].

In 1980's, the analysis of the response variability of stochastic systems received a lot of attention, consequently a new field, " Stochastic Finite Elements " was coined to stochastic mechanics. Although there have been papers on Monte Carlo solutions and reliability considerations, most of the studies done in stochastic finite elements have been on the second moment analysis of stochastic systems under deterministic loading.

An account for the state of the art in 1986 was given in a review article by Vanmarcke et al. [50] in which local averaging technique, Monte Carlo simulations and reliability considerations are stressed out. Following developments till 1988 were reviewed by Benaroya and Rehak [4]. In this review, available perturbation and linear partial derivative methods for
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second moment methods are discussed. Another good review in recent developments in finite elements, in which different discretization methods and reliability applications are outlined, is given by Der Kiureghian et al. [21]. Recently, an excellent update review has been provided in a report by Brenner [7]. Brenner gave a comprehensive overview of the basic research in static stochastic finite elements in a survey concerning theoretical backgrounds, strengths and weaknesses of various available stochastic finite element methods. The comparisons in this report are mainly focused on discretization schemes and future reliability applications.

As noted in these reviews, the evolution of stochastic finite element methods took place in the last decade. Handa and Anderson [28] employed first order perturbation method in connection with stochastic finite elements to solve simple static problems. Hisada and Nakagiri [30] incorporated second order perturbation method to static stochastic finite elements problems when structures have uncertain shapes. Vanmarcke and Grigoriu [52] used local averaging technique of Vanmarcke [49], a way of discretely representing random fields with the finite element method, to evaluate the second-order statistics of a beam with random rigidity under static loading. Liu et al. [35, 36, 37] developed an approximate second-order perturbation technique for linear and nonlinear static and dynamic problems. For dynamic problems, they observed good results in the mean, but significant deviations in the variance with time were present even if with very small variabilities of the random variables. In their example, the coefficient of variation for stiffness was taken as only 0.05. Lawrence [32] introduced basis random variable method in connection with Galerkin finite elements. The principle in this method is to expand the random fields as the sum of random variables multiplied by linearly independent and preferably orthogonal deterministic functions. Yamazaki, Shinozuka and Dasgupta [54] used Neumann expansion technique in static stochastic finite element problems to compare Neumann expansion technique with perturbation methods and Monte Carlo solutions. Spanos and Ghanem [43] proposed employing Karhunen-Loeve expansion for discretization of the random fields followed by Neumann expansion method for the solution of the static random equations. Later, Ghanem and Spanos [26] introduced the polynomial chaos concept to stochastic finite elements. In this method, the random response is simply expressed as a convergent series along an orthogonal polynomial basis. Der Kiureghian et al. [21] proposed a new discretization method named optimum linear estimation which is merely an interpolation method between nodal points where optimal deterministic interpolation functions are selected. Following the fundamentals of Liu et al. [35]'s work, Teigen et al. [47, 48] investigated nonlinear concrete structures under static random loads. Recently, Chang and Yang [8] developed a mean-centered approximate second-order perturbation method in conjunction with modal expansion to solve nonlinear dynamic problems.
with structural uncertainties for short durations. They employed equivalent linearization with Gaussian closure to treat nonlinearities and the local averaging method of Vanmarcke [49] to discretize random fields.

In all these aforementioned methods, the representation of the random fields by a small number of random variables is not exact and usually mesh dependent. The consequent limits on the finite-element size which depends on the correlation length of the stochastic field urges the analyst to use very fine meshes for shortly correlated fields. In a series of papers, Deodatis, Takada and Shinozuka introduced a new method in which Galerkin finite elements with deterministic shape functions is applied to stochastic differential equations. This new method named "Weighted Integrals" overcame the major drawbacks of the other methods, i.e. the results are mesh independent and random fields can be accurately represented by a small number of random variables of interest, Deodatis [13, 15, 17], Takada [44, 45, 46], Deodatis and Shinozuka [18]. There have also been some simple studies on stochastic shape functions, e.g. Dasgupta and Yip [12], Deodatis [15, 16].

The ultimate goal of stochastic response analysis is reliability considerations with limit states ranging from serviceability and applicability requirements to total collapse. In this respect, reliability of stochastic systems have been discussed by Der Kiureghian and Liu [19, 20], Liu and Der Kiureghian [34], Ghanem and Spanos [26], Lawrence et al. [33], Deodatis and Shinozuka [18], Der Kiureghian et al. [21]. In the mean time, spectral-distribution free bounds for the second-order moments of the response of stochastic systems are proposed by Shinozuka and Deodatis [42], Deodatis and Shinozuka [14], Deodatis [15, 16].

The motivation of the present study arises from the fact that the present methods of analysis for random structures subject to deterministic or random dynamic loadings are not satisfactory, even in the case of small variability of the random structural properties. As follows from the title of the paper, and in order to clearly emphasize the reasons for the shortcomings of the previous attempts, only linear SDOF systems are considered. However, the SDOF equation is assumed to be obtained from a modal expansion of a structural system retaining as an uncoupled term for each mode.

Mass, damping and stiffness of the SDOF system are assumed to be random functions of a finite set of basic variables $X$, defined from the discretization of the random structural field. The probabilistic structure of $X$, e.g. in terms of the joint probability density function, $f_X(x)$ is assumed to be known.

For a specific realization of $X = x$, the response of the SDOF system, $y(t; x)$, can be calculated. Then, the unconditional expectations $E[y^n(t; X)]$
can be obtained exactly using the total probability theorem.

\[
E[y^n(t; X)] = \int \ldots \int y^n(t; x)f_X(x)dx
\]  

where \( f_X(x) \) is the joint probability density function of vector \( X \).

Using second order Taylor’s expansion approximation for \( y(t; X) \) about the mean value \( x_0 = E[X] \) provides

\[
y(t; X) \approx y(t; x_0) + y_j(t; x_0)\Delta X_j + \frac{1}{2} y_{jk}(x_0)\Delta X_j\Delta X_k
\]  

where \( \Delta X_j = X_j - E[X_j] \), \( y_j(t; x_0) = \frac{\partial}{\partial X_j} y(t; x_0) \) and \( y_{ij}(t; x_0) = \frac{\partial^2}{\partial X_i \partial X_j} y(t; x_0) \). In equation (2), summation over the dummy indices must be performed. Following (2), it is straightforward to obtain the first and second order perturbed approximations for the mean value function, \( \mu_y(t) \), and the variance function, \( \sigma_y^2(t) \).

\[
\mu_y(t) \approx y(t; x_0) + \frac{1}{2} y_{jk}(t; x_0)C_{jk}
\]  

\[
\sigma_y^2(t) \approx y_j(t; x_0)y_k(t; x_0)C_{jk} + y_j(t; x_0)y_{kl}(t; x_0)E[\Delta X_j\Delta X_k\Delta X_l] + \frac{1}{4} y_{jk}(t; x_0)y_{lm}(t; x_0)(E[\Delta X_j\Delta X_k\Delta X_l\Delta X_m] - C_{jk}C_{lm})
\]

where \( C_{jk} = E[\Delta X_j\Delta X_k] \) signifies the components of the covariance matrix. First order perturbation solution for the mean and the variance consists of only the first terms of (3) and (4). Thus, for the mean values zeroth order perturbation solution, the solution for the mean deterministic system, is the same with the first order perturbed solution. If the third and fourth order terms are neglected in the variance, equation (4), the solution must be called approximate second order perturbation solution. Because, in this case, although the mean is second order accurate, the variance becomes first order accurate, e.g. Liu et al.’s work [36]. Hence, the variance of the first order solution and the approximate second order solution are the same.

Using Gaussian closure to evaluate the third and fourth order central moments simplifies equation (4) to

\[
\sigma_y^2(t) = y_j(t; x_0)y_k(t; x_0)C_{jk} + \frac{1}{2} y_{jk}(t; x_0)y_{lm}(t; x_0)C_{jl}C_{km}
\]
Along the text, exact solutions are compared with mean centered zeroth, first and second order perturbation solutions in several problems. The response of stochastic SDOF systems under eigenvibrations, harmonic loadings and horizontal earthquake random nonstationary earthquake loadings are studied in detail.

STOCHASTIC SDOF SYSTEMS UNDER EIGENVIBRATIONS

The equation of motion of a linear SDOF system with viscous damping has the following form.

\[
m\ddot{y} + c\dot{y} + ky = f(t) \hspace{1cm} y(0) = y_0 \hspace{1cm} \dot{y}(0) = \dot{y}_0
\]

where the combined stochastic variables \( m = m(X), c = c(X) \) and \( k = k(X) \) are respectively mass, viscous damping and the stiffness. \( y \) denotes the displacement of the SDOF system. \( y_0 \) and \( \dot{y}_0 \) are the initial values for the displacement and velocity, and, \( f(t) \) is the time dependent loading.

Omitting the explicit indication of the dependency on the structural system parameters, \( X \), equation (6) can be rewritten as

\[
m(\ddot{y} + 2\zeta\omega_0\dot{y} + \omega_0^2 y) = f(t)
\]

where

\[
\omega_0^2 = \frac{k}{m}, \hspace{1cm} \zeta = \frac{c}{2m\omega_0}
\]

\( \omega_0 \) denotes the undamped circular natural frequency and \( \zeta \) is the viscous damping ratio, characterizing damping as a nondimensional number.

The impulse response function (Green’s function) for the differential operator of equation (7) is

\[
h(t; X) = \begin{cases}
\frac{\exp(-\zeta\omega_0 t)}{m\omega_0 \sqrt{1-\zeta^2}} \sinh(\sqrt{\zeta^2 - \omega_0^2} t) & , \hspace{0.5cm} \zeta > 1, \hspace{0.5cm} t > 0 \\
\frac{1}{m} \exp(-\omega_0 t) & , \hspace{0.5cm} \zeta = 1, \hspace{0.5cm} t > 0 \\
\frac{\exp(-\zeta\omega_0 t)}{m\omega_0 \sqrt{1-\zeta^2}} \sin(\omega_0 \sqrt{1-\zeta^2} t) & , \hspace{0.5cm} \zeta < 1, \hspace{0.5cm} t > 0
\end{cases}
\]

The displacement can always be written in the integral form as

\[
y(t; X) = [ch(t) + m\dot{h}(t)]y_0 + mh(t)\dot{y}_0 + \int_0^t h(t - \tau)f(\tau)d\tau
\]

Consequently, the eigenvibrations, \( f(\tau) = 0 \), can be written as

\[
y(t; X) = [ch(t) + m\dot{h}(t)]y_0 + mh(t)\dot{y}_0
\]
Since for most of the civil engineering structures, damping ratio varies from 0.01 to 0.10, only the underdamped systems where $\zeta < 1$ is considered in this study. The Green's function for the underdamped SDOF systems in terms of the physical parameters of equation (6) is

$$h(t; X) = \exp\left(-\frac{ct}{2m}\right) \frac{\sin\left(\sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} t\right)}{\sqrt{km - \frac{c^2}{4}}}$$

In what follows, the Green's function of equation (12) is studied in detail. Because it is clear from equation (10) that response under any forcing function and any initial values for the displacement and velocity can easily be calculated using the Green's function.

Since $m$, $c$ and $k$ depend on the basic random variables of the system, $X$, the Green's function, $h(t; X)$, as defined by equation (12) becomes a stochastic function which is generated by $X$. Despite the linearity of the equation of motion, Green's function is a highly nonlinear function of the random quantities, so does the solution which is evaluated from the Duhamel's integral, equation (10).

In the case of eigenvibrations, the first order derivative of the displacement (11) with respect to $X_j$, $y_j(t; X)$, is proportional to $t$, the second derivative $y_{jk}(t; X)$ is proportional to $t^2$, etc., which imply a behaviour identical to secular terms in singular perturbation methods using the Taylor expansion of the response. This questions the application of all the classical first and second order perturbation techniques for the mean and the variance of the response, at least in the case of eigenvibration problems and external excitations with impulsive character. Actually, such defects have been observed by several researchers, see e.g. Frisch [23], Nakagiri et al. [39], Vanmarcke et al. [50], Liu et al. [36].

Figure 1. The exact $E[h(t, m, c, k)]$ and its first (zeroth) and second order perturbed approximations versus eigenperiod nondimensional time. $m = \mu_m = 1$, $c$ and $k$ are mutually independent parabolically distributed with, respectively, $PD(b_c = 0.10, \frac{\sigma_c}{\sqrt{5}} = 0.04)$ and $PD(b_k = 1.00, \frac{\sigma_k}{\sqrt{5}} = 0.20)$. 
As an example, consider the case \( f(r) = 0, \ y_0 = 0, \ \dot{y}_0 = \frac{1}{m}. \) Then \( y(t, m, c, k) = h(t, m, c, k). \) The mean value of the stochastic Green's function, \( E[h(t, m, c, k)] \), and the first and second order perturbed approximations of this quantity as a function of eigenperiod nondimensional time are plotted in Figure 1. For simplicity, \( m \) is assumed to be deterministic whereas \( c \) and \( k \) are modeled as mutually independent parabolically distributed random variables. Parabolic distribution as well as other probability distributions referred in this paper are summarized in Appendix 1. It should be noted that all the physical bounds of the problem, i.e. \( k > 0, \ c > 0, \ \zeta < 1, \) are fulfilled for all these probability distributions with probability 1.

The numerical values chosen for this example and for all the following examples in the text follow the legend under the figures. The computations for expectations at each time is carried out using Mathematica [53]. As shown in Figure 1, the first and second order perturbations for the mean of the stochastic Green's function differ sooner or later significantly from the exact solution. The second order approximation only postpones the time of deviation. The deviation of the perturbation solution gets worse in time. Therefore, the application of conventional mean centered perturbation methods for long durations is not able.

The zeroth order perturbation solution represents the response of the deterministic SDOF system with mean parameters, \( \mu_m, \ \mu_c, \ \mu_k. \) The fact that this solution is significantly different than the mean of the stochastic response, should convince engineers that stochastic analysis is definitely not only worthwhile but also essential.

![Figure 2](image)

**Figure 2.** The exact \( \text{Var}[h(t, m, c, k)] \), first order perturbation and second order perturbation with Gaussian closure of this quantity versus eigenperiod nondimensional time for the example of Figure 1.

For the same problem, exact variance of the response, \( \text{Var}[h(t, m, c, k)] \), its first and second order perturbations with Gaussian closure, equation (12), versus eigenperiod nondimensional time are plotted in Figure 2. Although good fits are observed for very short durations, both first and the second order perturbations with Gaussian closure blow up with time.
Figure 3. Random stiffness analysis. The exact $E[h(t, m, k)]$ and perturbation approximations versus eigenperiods. $m = \mu_m = 1$, $k$ is uniformly distributed in $[0.70, 1.30]$ with $\mu_k = 1.00$ and $\sigma_k = 0.1732$.

The difference in the amplitude of the curves in Figure 1 is not governed by the damping random term on the exponential multiplied with time. It is mainly due to the random stiffness term and the change in phase is due to the random damped frequency. In order to clearly show these effects, an analytical solution for the mean value of an undamped stochastic Green’s function $h(t, m, k)$ is derived. For simplicity, $k$, or equivalently $\omega_0^2$, is considered as a random variable uniformly distributed in the interval $[\mu_k - \sqrt{3}\sigma_k, \mu_k + \sqrt{3}\sigma_k]$, where $\mu_k = E[k]$ and $\sigma_k^2 = Var[k]$. With $m = 1$, it can be shown that

$$E[h(t, m, k)] = \frac{1}{\sqrt{3}\sigma_k t} \left( \cos \left( \sqrt{\mu_k - \sqrt{3}\sigma_k} t \right) - \cos \left( \sqrt{\mu_k + \sqrt{3}\sigma_k} t \right) \right)$$ (13)

This exact solution and perturbation solutions versus eigenperiod nondimensional time are plotted in Figure 3. For the present undamped case, the zeroth order solution is an undamped sine curve with a period determined by the stiffness $\mu_k$. As shown in (13), the exact solution will damp out inversely with time. This effect is present for all the considered distributions, see Fig. 5.a, and will be termed stochastic artificial damping.

Figure 4. Random stiffness analysis. The exact $Var[h(t, m, k)]$ and perturbation approximations versus eigenperiod for the example of Fig. 3.
For the same example, the variance of the Green's function is plotted as a function of eigenperiod nondimensional time in Figure 4. Exact variance stabilizes after some oscillations and the perturbations are meaningless after a few periods. First order perturbation and Gaussian closure results diverge with second order and fourth order errors, respectively.

Figure 5.a ) Only random stiffness analysis. Probability density function dependency of $E[h(t, m, k)]$. Figure 5.b ) Probability density function dependency of $\text{Var}[h(t, m, k)]$. $m = \mu_m = 1$, $k$ is a, respectively, uniformly, parabolically and triangularly distributed random variable with $\mu_k = 1$ and $\sigma_k = 0.1732$.

Figures 5.a and 5.b show the corresponding results for Figure 3 and Figure 4 in which the mean and the variance of $k$ is kept constant while the probability density function (pdf) assigned to it is altered. These results show that the mean and the variance of $h(t, m, k)$ is not strongly dependent on the pdf of $k$ beyond the second moment properties. Hence, a better approach for the solution of any transient vibration problem than using second moment based perturbation methods is to assign an engineering-wise acceptable pdf to $k$ with mean $\mu_k$ and standard deviation $\sigma_k$ and to carry out the analysis accordingly to obtain a closed form solution such as (13).

Figure 6. Random damping analysis. Exact $E[h(t, m, c, k)]$ and its perturbation approximations versus eigenperiods. $m = \mu_m = 1$, $k = \mu_k = 1$ and $c$ is parabolically distributed with $PD(\mu_c = 0.10, \frac{\sigma_c}{\sqrt{5}} = 0.04)$.

In order to investigate the random damping effect, a similar analysis for
damped impulse response function, \( h(t, m, c, k) \), is worked out in Figure 6. Stiffness and mass are kept deterministic and only damping is assumed to be a parabolically distributed random variable. In this case, the arising secular terms of perturbation solutions are under the governing control of the exponential decay, thus, very good results are observed for the perturbed solutions. For lightly damped systems, the secular terms of the perturbation approximations only become significant after the dissipation of the eigen vibrations. Hence, perturbation solutions will never diverge and are very good approximations for lightly damped systems. In order to show the governing control of the exponential decay, respectively an exact solution and its second order approximation are given analytically in (14) and (15) assuming \( c \) to be uniformly distributed in \( [\mu_c - \sqrt{3}\sigma_c, \mu_c + \sqrt{3}\sigma_c] \).

\[
E[e^{-ct}] = \frac{1}{2\sqrt{3}\sigma_c t} e^{-\mu_c t} \left( e^{\sqrt{3}\sigma_c t} - e^{-\sqrt{3}\sigma_c t} \right) \tag{14}
\]

\[
E[e^{-ct}] \approx e^{-\mu_c t} \left( 1 + \frac{t^2\sigma_c^2}{2} \right) \tag{15}
\]

In this case, second order perturbation underestimates the exact result. For \( \mu_c = 2\sigma_c \) and \( \sigma_c t = 1 \), approximately 5.1 percent error is present after the eigenvibration has decayed to one fifth, thus, the absolute difference between the exact solution and the second order perturbed solution is approximately 1 percent of the amplitude of the initial eigenvibration. Although the error slowly tends to grow up to 100 percent as \( t \to \infty \), because of the exponential decay term, (14) and (15) diminish faster. Therefore, the differences are neither observable nor important. Similar results are also valid for the second order statistics of the response.

In conclusion, the present analysis has shown that for eigenvibrations, stochastic response of the SDOF with random stiffness cannot be determined using perturbation methods due to secular terms in the expansion. Extensions to third and higher order will not improve the solution. On the contrary, secular terms of the corresponding order will arise, thus the solution will blow out faster. When only the damping coefficient is random, perturbation solutions are seemed to be very good approximations.

STOCHASTIC SDOF SYSTEMS UNDER HARMONIC LOADING

The previous section dealt with the response due to initial values and impulsive type of loadings. In this section, the opposite limit is studied, where the transient effects have decayed and stationary vibrations have been attained.

Under the harmonic excitation, \( f(\tau) \)

\[
f(t) = A\cos(\Omega t) \tag{16}
\]
with a deterministic circular frequency $\Omega$ and a deterministic amplitude $A$, the stationary response of the SDOF system for each realization becomes harmonic with the same circular frequency $\Omega$.

$$y(t; m, c, k) = |H(\Omega)|A \cos(\Omega t - \psi)$$

(17)

where

$$|H(\omega)| = \frac{1}{\sqrt{(\Omega^2 - \omega^2)^2 + 4\zeta^2\omega^2\Omega^2}}$$

(18)

$$\tan \psi = \frac{2\zeta\omega_0\Omega}{\omega_0^2 - \Omega^2}$$

(19)

In this case, the partial derivatives of $y(t; m, c, k)$ with respect to $m$, $c$ and $k$ of any order will not contain any secular terms. Hence, there is no tendency that the first and second order perturbation solutions eventually deviate arbitrarily from the exact solution, as in the transient case.

Figure 7. Stationary harmonic vibrations. Exact $E[y(t, m, c, k)]$ and its perturbation approximations versus eigenperiods. $\Omega = 2$, $m = \mu_m = 1$, $c = \mu_c = 0.10$ and $k$ is uniform in $[0.70, 1.30]$ with $\mu_k = 1.00$ and $\sigma_k = 0.1732$.

This has been shown in Figure 7 for $\Omega = 2\sqrt{E[k]}$. Only $k$ is taken as a random variable with uniform distribution. The exact solution, first and the second order approximations will be harmonic functions with same circular frequency with slightly different phases and amplitudes. In the present case, well away from the expected resonance range, $\Omega \simeq \sqrt{E[k]}$, the random phase is very small for all realizations of the structure, and no visible phase difference between the solutions are observed. However, in the expected resonance range, substantial phase differences may be present along with large differences in the amplitudes of the perturbed approximations.

Any non-impulsive external loading can be expressed in terms of a possible infinite sum of harmonic terms through its Fourier transform. For each
of these harmonic terms, the previous analysis is valid. Then, it can be stated that the first and second order perturbation analysis can be applied without significant errors for the stationary response where any response from the initial values has been dissipated.

The presented discussions on stochastic eigenvibrations and stochastic stationary harmonic vibrations clearly explain the reasons for the numerical observations of Liu et al. [36], Nakagiri et al. [39], Vanmarcke et al. [50].

**STOCHASTIC SDOF SYSTEMS UNDER HORIZONTAL EARTHQUAKE EXCITATIONS**

Horizontal ground earthquake accelerations, $\ddot{X}_g$, are zero-mean nonstationary random processes which are mostly modeled as time modulated stationary stochastic processes.

\[ \ddot{X}_g = A(t)B(t) \]  

(20)

where $A(t)$ is a deterministic modulation function and $B(t)$ is a stationary zero-mean stochastic process.

Next, an analytically tractable example is worked out to calculate the time dependent variance for the displacement of an oscillator subject to a time modulated white-noise excitation. Let $B(t) = W(t)$ denote unit white noise and $y(t)$ signify the horizontal displacement of the mass relative to the ground surface. Consequently, the excitation force $f(t)$ becomes $-mA(t)W(t)$. Under this loading the conditional variance of the displacement as a function of time, for each realization of the oscillator, with zero initial conditions can be evaluated from (5) as

\[ \sigma_y^2(t) = m^2 \int_0^t \int_0^t h(t - \tau)h(t - \bar{\tau})A(\tau)A(\bar{\tau})E[W(\tau)W(\bar{\tau})]d\tau d\bar{\tau} \]  

(21)

For unit white noise

\[ E[W(\tau)W(\bar{\tau})] = \delta(\tau - \bar{\tau}) \]  

(22)

Substitution of (22) into (21) yields

\[ \sigma_y^2(t) = m^2 \int_0^t [h(t - \tau)A(\tau)]^2 d\tau \]  

(23)

For many forms of time envelope, $A(t)$, this integral can be evaluated analytically, e.g. a sum of two exponentials fit.

\[ A(t) = \alpha(e^{-\beta t} - e^{-\gamma t}) \]  

(24)
where duration and the strength of the earthquake is defined by the envelope parameters, $\alpha$, $\beta$ and $\gamma$. In what follows, the random stiffness and random damping affects are investigated separately using $A(t)$ in the form of (24). The energy content of an earthquake is proportional to $\int_0^\infty A(t)^2 dt$.

Keeping this as constant and altering the envelope parameters $\alpha$, $\beta$, $\gamma$ is tantamount to studying different types of earthquakes in duration and strength with the same energy content. Three different envelope functions with the same energy content are assigned to represent, respectively, very short duration, El-Centro and very long duration earthquakes, see the right corner of Figure 8.

Figure 8. The unconditional variance $E[\sigma_y^2(t)]$ for the stochastic SDOF system with $m = \mu_m = 1$, $c$ and $k$ uniformly distributed in $[0.03, 0.07]$ and $[0.7, 1.3]$, respectively, subject to El-Centro Earthquake, $A$, $(\alpha = 2.32, \beta = 0.09, \gamma = 1.49)$. 
Figure 9. The unconditional variance $E[\sigma_y^2(t)]$ of the same stochastic SDOF system of Figure 8 subject to a short duration earthquake, $B$, ($\alpha = 533$, $\beta = 1.9$, $\gamma = 1.95$).

Figure 10. The unconditional variance $E[\sigma_y^2(t)]$ for the same SDOF system of Figure 8 subject to a longer duration earthquake, $C$, ($\alpha = 0.7109$, $\beta = 0.01$, $\gamma = 2.00$).

The corresponding exact unconditional variance for the stochastic oscillator and its first order perturbed approximation are plotted as a function of eigenperiods in Figure 8, Figure 9 and Figure 10. It is clear from the figures that in the case of only random damping, perturbation solution is very accurate for all three kinds of earthquakes. However, as shown in Figure 9, for only random stiffness, due to the secular terms, perturbation solutions are not accurate for short duration earthquakes. The effect of
stochastic artificial damping can be seen clearly in Figure 9. Although it is not shown, as they carry higher order secular terms, second order perturbations diverge faster. On the other hand, according to the studied cases, perturbation solutions seem to be good in estimating the maximum variance of the SDOF system in any case.

CONCLUSION

The dynamic response of linear SDOF systems with random mass, damping and/or stiffness subject to deterministic and random excitations has been studied. The conventional zeroth, first and second order perturbation solutions are compared to the exact solution which is evaluated applying the total probability theorem to the conditional results derived for a specific realization of the random variables.

Random damping and random stiffness effects are investigated separately considering two limit behaviours of oscillators; eigenvibrations and stationary response under harmonic loadings. For eigenvibrations, in both cases, perturbation solutions carry divergent secular terms. When only random stiffness is considered, due to the secular divergent terms, perturbation solutions blow up with time. In the case of only random damping, since the divergent secular terms are under the governing control of the exponential decay, the existing deviations in the perturbed solutions become neither observable nor important. Hence, perturbation solutions don't diverge from the exact solution and are seemed to be very good approximations. Under harmonic loading, after the dissipation of eigen vibrations, perturbation solutions don't possess any secular terms and are very good approximations except for the case in which the system is excited at or close to its resonance frequency.

Finally, short duration nonstationary random response of stochastic SDOF systems subject to random seismic excitations are investigated in detail. It is observed that perturbation methods are good approximations in the case of random damping. When only stiffness is considered random, due to the secular terms, perturbation solutions are poor approximations for short duration earthquakes, yet seem to be useful in estimating the maximum variance of the SDOF system in any case.

In summary, it is concluded that perturbation solutions for dynamic problems can be used in the case of random damping. In the case of random stiffness, due to the secular terms in eigenvibrations, perturbations solutions may significantly diverge from the exact solution with time, even in the case of small variabilities of random variables.

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REFERENCES


Soil Dynamics and Earthquake Engineering


APPENDIX

Uniform Distribution, $U(a, b)$

$$f_X(x) = \frac{1}{b-a}, \quad a \leq x \leq b$$  \hspace{1cm} (A.1)

$$E[X] = \frac{a+b}{2}$$  \hspace{1cm} (A.2)

$$Var[X] = \frac{(b-a)^2}{12}$$  \hspace{1cm} (A.3)

Symmetric Triangular Distribution, $TD(b, \frac{a}{\sqrt{6}})$

$$f_X(x) = \begin{cases} \frac{1}{a^2}(x-b) + \frac{1}{a}, & b-a \leq x \leq b \\ \frac{1}{a^2}(x-b) + \frac{1}{a}, & b \leq x \leq b + a \end{cases}$$  \hspace{1cm} (A.4)

$$E[X] = b$$  \hspace{1cm} (A.5)

$$Var[X] = \frac{a^2}{6}$$  \hspace{1cm} (A.6)

Symmetric Parabolic Distribution, $PD(b, \frac{a}{\sqrt{5}})$

$$f_X(x) = \frac{3}{4a^3}(-x^2 + 2bx - b^2 + a^2), \quad b-a \leq x \leq b + a$$  \hspace{1cm} (A.7)

$$E[X] = b$$  \hspace{1cm} (A.8)

$$Var[X] = \frac{a^2}{5}$$  \hspace{1cm} (A.9)