Analysis of vertical vibration of earth dam in triangular canyons
X. Zhiying

Department of Irrigation and Drainage Engineering, Hohai University, Nanjing, China

ABSTRACT

In this paper, firstly, a partial differential equation of vertical motion for an earth dam with symmetrical cross-section in a triangular canyon is presented. Then by using the method of separation of variables and Bubnov-Galerkin approach, an approximate eigenvalue solution for the fundamental natural frequency of vertical vibration of the dam is obtained. An approximate calculation formulas for vertical earthquake response of dam are given. These formulas are very simple and dynamic analysis may readily be made by hand calculation. At the end, a sample calculation has been presented. Some interesting and reasonable results have been obtained.

INTRODUCTION

Designing an earth and rockfill dam to resist earthquake damage is probably one of the most difficult tasks to be faced by the geotechnical and earthquake engineer. Current methods for determining the seismic stability of an earth and rockfill dam usually involve a dynamic response analysis of the dam for the maximum earthquake motions to affect the structure. Generally, the assumption of plane strain condition for analyzing the dynamic response of dam is adopt-
ed. If earth and rockfill dam is built in a rectangular canyon where the width of the canyon is large with respect to the height of dam, this assumption is considered an adequate representation. However, for cases of narrow rectangular canyons or triangular canyons, the response of a dam is greatly affected by the proximity of the rigid boundaries, and the validity of plane strain assumption becomes questionable. In such cases, a three-dimensional solution is desirable to predict accurately the dynamic response of an embankment. Probably the most versatile tool currently available to perform such analyses is the 3-D FEM [1-3], but it would be very costly.

Existing analytical techniques for earth and rockfill dam essentially assume the response analysis restricted to horizontal shear deformation in the upstream-downstream direction. Due to these restrictive assumptions, the dynamic response analysis cannot be used to examine the nature of stress distribution within an earth dam due to vertical ground motion. However, the strong ground motion in the vertical direction is often observed at epicentral areas and their vicinities. The maximum accelerations of these motions frequently reach the levels of maximum accelerations in the horizontal direction. Thus the strong ground motion in the vertical direction will play an important role in the stability of earth dams, and the problem of earthquake-induced vertical vibration of earth dams is significant for the dam's safety but has received comparatively little attention. This paper develops an approximate analytical method for evaluating the dynamic responses of earth dams in the vertical direction.

EQUATION OF VERTICAL VIBRATION OF EARTH DAM

Fig. 1 shows a longitudinal section, a transversal section and a transversal slice of dam, on which vertical forces are acting under vertical motion. Assume longitudinal section and transversal section are symmetrical triangle. The assumptions inherent to a shear wedge analysis of a symmetrical earth dam are as follows: (1) The canyon wall is perfectly rigid; (2) The direction of ground motion is vertical and there is no displacement in other directions; (3) The dam is homogeneous and the dam materials are linearly elastic; (4) Interaction between water in the reservoir and the dam is negligible.
Forces acting on an element in the vertical direction as shown in Fig. 1 (c) are:

1. Inertial force

\[ F_i = \rho (l + \frac{1}{2} \frac{\partial l}{\partial y} dy)dydz \frac{\partial^2 v}{\partial t^2} \]

2. Shear force on front

\[ S_w = G \frac{\partial v}{\partial z} (l + \frac{1}{2} \frac{\partial l}{\partial y} dy) dy \]

3. Shear force on back face

\[ S_w + d(S_w) = G \left( \frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial z^2} dz \right) (l + \frac{1}{2} \frac{\partial l}{\partial y} dy) dy \]

4. Normal force on bottom

\[ F_r = \sigma_{r} ldz = lE \frac{\partial v}{\partial y} dz \]

5. Normal force on top

\[ F_r + d(F_r) = E \left( \frac{\partial v}{\partial y} + \frac{\partial^2 v}{\partial y^2} dy \right) (l + \frac{\partial l}{\partial y} dy) dz \]

Where \( v \) is displacement in \( y \) direction, \( \rho \) density of material, \( l \) the width of element in \( x \) direction, \( \sigma_{r} \) normal stress in \( y \) direction, \( G \) shear modulus of the material, \( E \) the elastic modulus of the material and \( t \) time. \( E = 2(1 + \mu) G = \xi G \), in which \( \mu \) is the Poisson's ratio of material and \( \xi = 2(1 + \mu) \).

As shown in Fig. 1 (c), under undamped condition, two shear forces, two normal forces and one inertia force act on each element. Dynamic equilibrium of element requires that:

\[ G \xi \frac{\partial^2 v}{\partial y^2} + G \frac{\partial^2 v}{\partial z^2} + \xi \frac{G}{l} \frac{\partial v}{\partial y} - \rho \frac{\partial^2 v}{\partial t^2} = 0 \]

(1)
Figure 1: Analytical model of dam in triangular canyon for analysis.

From geometrical consideration it follows that:

\[ \frac{1}{l} \frac{\partial}{\partial y} = \frac{1}{y - H} \]  \hspace{1cm} (2)

Substituting equation (2) into equation (1), the equation of motion governing free vertical vibration of dam is obtained:

\[ v^2 \frac{\partial^2 v}{\partial y^2} + v^2 \frac{\partial^2 v}{\partial z^2} + \frac{v^2 \xi}{y - H} \frac{\partial v}{\partial y} - \frac{\partial^2 v}{\partial y^2} = 0 \]  \hspace{1cm} (3)

The following boundary conditions are applicable to the case of symmetrical dam in a triangular canyon:

\[ \begin{align*}
\frac{\partial v}{\partial y} &= 0 \quad \text{at} \quad y = H \\
v &= 0 \quad \text{at} \quad y = \frac{2H}{L}z = Kz \\
\frac{\partial v}{\partial z} &= 0 \quad \text{at} \quad z = 0
\end{align*} \]  \hspace{1cm} (4)
where $L$ is length of dam crest.

**SOLUTION FOR FIRST NATURAL FREQUENCY**

By the method of separation of variable $\left[ v(\ y, \ z, \ t) = \psi(\ y, \ z) \ T(\ t) \right]$, the following equations are obtained:

\[
\frac{\partial^2 T}{\partial y^2} + \omega^2 T = 0 \tag{5}
\]

\[
\zeta \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{\zeta}{y - H} \frac{\partial \psi}{\partial y} + \frac{\omega^2 \psi}{v_z^2} = 0 \tag{6}
\]

where $\omega$ is the natural frequency. Therefore:

\[
T = C_1 \cos \omega t + C_2 \sin \omega t \tag{7}
\]

where $C_1$ and $C_2$ are arbitrary constants.

Since the boundary conditions given by equation (4) have to be satisfied at all times, the following boundary condition can be imposed on the function $\psi$:

\[
\begin{align*}
\frac{\partial \psi}{\partial y} &= 0 \quad \text{at} \quad y = H \\
\psi &= 0 \quad \text{at} \quad y = \frac{2H}{L} z = Kz \\
\frac{\partial \psi}{\partial z} &= 0 \quad \text{at} \quad z = 0
\end{align*} \tag{8}
\]

Solution in closed form of equation (6) is difficult to obtain. However, an approximate eigenvalue solution of equation (6) can easily be used to obtain a rather accurate value for the first natural frequency of vibration of the system.

According to the Bubnov-Galerkin method, if a function $\psi$ which satisfies the boundary conditions given by equation (8) can be found, the following integral:

\[
\int_0^H \int_0^{y/K} \left( \zeta \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{\zeta}{y - H} \frac{\partial \psi}{\partial y} + \frac{\omega^2 \psi}{v_z^2} \right) y \, dz \, dy = 0 \tag{9}
\]

yields an algebraic equation from which the frequency of the system can be determined.

It can easily be shown that the function:

\[
\psi = \frac{1}{H} (y + Kz)(y - Kz)(y - 2H + Kz)(y - 2H - Kz) \tag{10}
\]
satisfies the equation (8). After substituting equation (10) into equation (9) and performing the integration the following algebraic equation is obtained:

$$\frac{32}{225} \frac{\omega^2 H^2}{K} \frac{8}{v^2} - \frac{32}{45} K = 0$$ (11)

Solving equation (11) for $\omega$, we get:

$$\omega = \frac{v_x}{H} \sqrt{\frac{45}{4} \xi + 5K^2}$$ (12)

This expression gives the first natural frequency, i.e. $\omega_1$ of a symmetrical dam in a triangular canyon under vertical vibration, the function $\psi$ in equation (10) is corresponded to the first mode shape of vibration, i.e. $\psi_1$.

EARTHQUAKE RESPONSE ANALYSIS OF DAM

It is easily proved that the equation governing vertical vibration of dam with damping under earthquake can be written as:

$$\frac{\partial^2 v}{\partial t^2} + \frac{c}{\rho} \frac{\partial^2 v}{\partial t} - \frac{G}{\rho} (\xi \frac{\partial^2 v}{\partial y^2} + \frac{\xi}{y - H} \frac{\partial v}{\partial y}) = -\ddot{v}_y(t)$$ (13)

Where $\ddot{v}_y(t)$ is acceleration of rigid canyon in the $y$ direction and $c$ is coefficient of damping.

By the method of separation of variables $[v(y,z,t) = \sum_{l=1}^{\infty} \psi_l(y,z) T_l(t)]$ and based upon the orthogonality of mode shape, the following two equations for the first mode shape are obtained:

$$\xi \frac{\partial^2 \psi_1}{\partial y^2} + \frac{\partial^2 \psi_1}{\partial z^2} + \frac{\xi}{y - H} \frac{\partial \psi_1}{\partial y} + \frac{\omega_1^2}{v_x^2} \psi_1 = 0$$ (14)

$$\frac{\partial^2 T_1}{\partial \xi^2} + 2 \omega_1 \omega_1 \frac{\partial T_1}{\partial \xi} + \omega_1^2 T_1 = -\eta_1 \ddot{v}_y(t)$$ (15)

where $\omega_1$ is first natural frequency given by equation (12), $\lambda_1$ is damping ratio of first mode, it is equal to $c/2\rho \omega_1$, $\eta_1$ is mode participate coefficient

$$\eta_1 = \frac{\int_0^H \int_0^{\psi_1} \psi_1 (H - y) dy dz}{\int_0^H \int_0^{\psi_1} \psi_1^2 (H - y) dy dz}$$
The solution of equation (15) is

\[
T_1 = \frac{-1.839}{\omega'_1} \int_0^t \tilde{v}_y(\tau)e^{-\omega_1(t-\tau)}\sin\omega'_1(t-\tau) \, d\tau
\]  

(16)

where \(\omega'_1 = \omega_1 \sqrt{1 - \lambda_1^2}\), the Duhamal integral may be calculated by numerical integration method.

Because the higher modes have little effect on earthquake response of dam, only a few lower modes (1 ~ 3 order) are adopted for practical requirement. Then the earthquake responses of dam in triangular canyons can be approximately written as follows (In order to simplify, the subscripts “1” of symbol \(\omega_1\), \(\omega'_1\), \(\eta_1\), \(\lambda_1\), \(\psi_1\) and \(T_1\) in the following formulas are omitted):

1. Displacement response

\[
v \approx \frac{1}{H^2}(y - Kz)(y + Kz)(y - 2H + Kz)(y - 2H - Kz) \times
\]

\[
-\frac{\eta}{\omega'_1} \int_0^t \tilde{v}_y(\tau)e^{-i\omega(t-\tau)}\sin\omega'(t-\tau) \, d\tau
\]  

(17)

2. Velocity response

\[
\ddot{v} \approx \frac{1}{H^2}(y - Kz)(y + Kz)(y - 2H + Kz)(y - 2H - Kz) \times
\]

\[
\; -\frac{\eta}{\omega'_1} \int_0^t \tilde{v}_y(\tau)e^{-i\omega(t-\tau)}\cos\omega'(t-\tau) \, d\tau - \lambda_1 \omega T
\]  

(18)

3. Acceleration response

\[
\ddot{v} \approx -\ddot{v}_y(t) + \frac{1}{H^2}(y - Kz)(y + Kz)(y - 2H + Kz)(y - 2H - Kz)\eta \times
\]

\[
\left[ \frac{(1 - 2\lambda^2)\omega_1}{\sqrt{1 - \lambda^2}} \right] \int_0^t \tilde{v}_y(\tau)e^{-i\omega(t-\tau)}\sin\omega'(t-\tau) \, d\tau +
\]

\[
2\lambda_1 \omega \int_0^t \tilde{v}_y(\tau)e^{-i\omega(t-\tau)}\cos\omega'(t-\tau) \, d\tau
\]  

(19)

4. Shear stress response
\[ \tau_{y} = \frac{G}{H^4} (4K^4z^3 - 4K^2yz + 8HK^2yz - 8H^2K^2z) \times \]
\[ \frac{-\eta}{\omega} \int_{0}^{t} \ddot{\psi}_{y}(\tau)e^{-i\omega(t-\tau)} \sin \omega(t - \tau) d\tau \] (20)

5 Vertical normal stress response

\[ \sigma_{y} = \frac{E}{H^4} (4y^3 - 12Hy^2 + 8H^2y - 4K^2z^2y + 4HK^2z^2) \times \]
\[ \frac{-\eta}{\omega} \int_{0}^{t} \ddot{\psi}_{y}(\tau)e^{-i\omega(t-\tau)} \sin \omega(t - \tau) d\tau \] (21)

In engineering it is most interesting in the max. response of dam and so the following formulas of max. response are useful for earthquake-resistant design of dam:

\[ v_{\text{max}} \approx |\psi\eta|_{S_{d}} \approx \left| \frac{1.839}{H^4} (y + Kz)(y - Kz) \right| S_{d} \] (22)

\[ \dot{v}_{\text{max}} \approx |\psi\eta|_{S_{v}} \approx \left| \frac{1.839}{H^4} (y + Kz)(y - Kz) \right| S_{v} \] (23)

\[ \ddot{v}_{\text{max}} \approx |\psi\eta|_{S_{a}} \approx \left| \frac{1.839}{H^4} (y + Kz)(y - Kz) \right| S_{a} \] (24)

\[ \tau_{y,\text{max}} \approx G|\psi\eta|_{S_{d}} \approx G \left| \frac{1.839}{H^4} (4K^4z^3 - 4K^2yz + 8HK^2yz - 8H^2K^2z) \right| S_{d} \] (25)

\[ \sigma_{y,\text{max}} \approx E|\psi\eta|_{S_{d}} \approx E \left| \frac{1.839}{H^4} (4y^3 - 12Hy^2 + 8H^2y - 4K^2z^2y + 4HK^2z^2) \right| S_{d} \] (26)

where \( S_{d}, S_{v}, \) and \( S_{a} \) are displacement response spectrum, velocity response spectrum and acceleration response spectrum respectively, as shown in Fig. 2 [4].

COMPUTED EXAMPLE

Suppose the symmetrical earth dam in the triangular canyon is subjected a vertical earthquake (EL Centro record in 1940), the max. height of dam \( H = 50 \text{m} \), the dynamic properties of dam material are \( v_{s} = 200 \text{m/s} \), damping ratio \( \lambda = 0.1 \), Poisson's ratio \( \mu = 0.3 \), Determine the first natural frequencies and the various max. responses in the central and L/4 sections of dam with the charts on
Figure 2: El Centro earthquake (1940) response spectrum
(a) Velocity spectrum; (b) Acceleration spectrum; (c) Displacement spectrum.
Figure 3: Distribution of various max. responses
(a) ~ (e) — central section;  (f) ~ (j) — L/4 section
1, 2, 3, 4, 5 — case in Table 1
Soil Dynamics and Earthquake Engineering

Fig. 2, for the cases of \( L=50\text{m}, 100\text{m}, 200\text{m} \) and \( 300\text{m} \) (i.e. \( L/H=1, 2, 4, 6 \)) respectively.

**Solution:** For the case of \( L=50\text{m} \), \( K=2H/L=2\times50/50=2 \). Substituting the values of \( v, H \) and \( K \) into equation (12), we get the first natural frequency \( \omega \) and first natural period \( T_p \) as follows:

\[
\omega = \frac{200}{50} \sqrt{\frac{45}{4} \times 2(1 + 0.3) + 5 \times 2^2} = 28.08/\text{s}
\]

\[
T_p = \frac{2\pi}{\omega} = \frac{2 \times 3.14}{28.08} = 0.22\text{s}
\]

According to this value of \( T_p \) and from the charts of response spectrum we get:

\( S_0 = 15\text{cm}, S_1 = 0.6\text{cm}, S_2 = 350\text{cm}/\text{s}^2 \)

Substituting the value of \( S_0, S_1, S_2, \eta \) and \( G \) into equations (22) — (26), the various max. response distributions with depth are obtained. For the case of \( L=100\text{m}, 200\text{m} \) and \( 300\text{m} \), an analogous computation has been performed. The computed results are summarized as in Table 1 and Fig. 3.

<table>
<thead>
<tr>
<th>Table 1 Computed Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>case No.</td>
</tr>
<tr>
<td>----------</td>
</tr>
<tr>
<td>( L ) (m)</td>
</tr>
<tr>
<td>( L/H )</td>
</tr>
<tr>
<td>( \omega ) (1/s)</td>
</tr>
<tr>
<td>( T_p ) (s)</td>
</tr>
</tbody>
</table>

**CONCLUSION**

The approximately analytical formulas developed in this paper are very simple and they can be used for analysis of vertical vibration of earth and rock-fill dam in triangular canyons under earthquake notion, by corresponding simplified method, such as response spectrum technique. Generally, analyses may readily be made by hand calculation even without a computer. A detailed example computation has shown that the analytical model presented here will provide information of practical as well as academic significance. The spatial distributions of various max. response are conforming with general law.

**REFERENCES**

