Application of a spectral element method to wave propagation in multiphase porous media

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Abstract

In this paper, Sneulders’ modification of Biot’s saturated poroelastic theory is used to account for the presence of a small amount of air in the pores of an unsaturated medium. The influence of the gas phase on the compressibility of the generalized pore fluid is accounted for by a complex effective bulk modulus of the gas phase. This modification is introduced in a spectral element formulation, enabling to model transient wave propagation in layered dry, saturated and unsaturated poroelastic media. The extended application area of the method is illustrated by a numerical example.

1 Introduction

A lot of materials encountered in civil, geophysical and biomechanical engineering can be considered as a porous medium consisting of an assemblage of solid particles and a pore space. The pore space may be filled with air (dry medium), a fluid (saturated medium) or a fluid and a small amount of air (unsaturated medium). The pore space is connected, enabling the filtration of the pore fluid through the porous medium. The solid skeleton is built up by the solid matrix and the empty connected pore space. Non-connected pores in the solid matrix may contribute to an additional compressibility of the solid matrix.

The aim of this paper is to present a unified approach to calculate wave propagation in horizontally layered dry, saturated and unsaturated poroelastic media. A considerable effort has been reported on the calculation of the dispersion characteristics (Thomson [19] and Haskell [9]) and the propagation of transient waves (Kausel and Roesset [11], Luco and Apsel [12] and Apsel and Luco [1]) in dry multilayered media. Using Biot’s poroelastic theory [2], Deresiewicz [7] and Jones [10] have studied the propagation of
free surface waves in a saturated poroelastic halfspace, while Paul [13] and Philippacopoulos [15, 17] have considered transient waves. Philippacopoulos [14, 16] has been the first to study the case of a ‘partially’ saturated poroelastic halfspace where a dry layer is on top of a saturated halfspace, accounting for the presence of a moving water table. Degrande [4] and Degrande and De Roeck [5] have presented a spectral layer and throw-off element to study wave propagation in multilayered dry and saturated poroelastic media.

In this paper, we follow Smeluders [18] and adapt Biot’s poroelastic theory to account for the presence of a small amount of air in the pores of a saturated poroelastic medium. The extended applicability of the spectral element formulation is demonstrated in a numerical example, where the influence of a moving water table and partial saturation on the wave propagation in an axisymmetric halfspace is studied.

2 Governing equations

The displacement vectors in both phases of a saturated porous medium are denoted by $\mathbf{u}^\alpha$ where $\alpha = s$ for the solid skeleton and $\alpha = f$ for the pore fluid. The fluid flow relative to the solid skeleton measured in terms of volume per unit area of the bulk medium is equal to $\mathbf{w} = n(u^f - u^s)$ where $n$ is the porosity. $\varepsilon^s = 1/2 \left[ \text{grad } \mathbf{u}^s + (\text{grad } \mathbf{u}^s)^T \right]$ denotes the small strain tensor in the solid skeleton while $\zeta = \text{div } \mathbf{w}$ is the volume of fluid which escapes from the pores of a unit volume of bulk material. The global equilibrium equation of the saturated porous medium is:

$$\text{div } \mathbf{t} + \rho \mathbf{b} = \rho \ddot{\mathbf{u}}^s + \rho_f \ddot{\mathbf{w}}$$

$t$ is the total stress tensor, $\rho \mathbf{b}$ is the body force, $\rho = \rho_f n + \rho_s (1 - n)$ is the density of the frozen mixture and $\rho_f$ is the density of the pore fluid. If the connected pore space is empty ($\rho_f = 0$), equation (1) reduces to the single phase equilibrium equation of classical continuum mechanics. The motion of the pore fluid with respect to the solid skeleton can be described by a generalized Darcy law (Biot [2], Coussy [3]):

$$\text{grad } \pi_p + \rho_f \mathbf{b} = \rho_f \ddot{\mathbf{u}}^s + \frac{\rho_f n}{n} \dot{\mathbf{w}} + \xi \mathbf{w}$$

$\pi_p$ is the pore fluid stress. The scalar correction factor $a > 1$ accounts for the tortuosity of the pores. $\xi$ is a symmetric positive definite second order tensor, directly related to the Darcy permeability tensor $k$ or the intrinsic permeability tensor $k_f$: $\xi = n^2 \gamma_f F k^{-1} = n^2 \nu_d F k_f^{-1}$ where $\gamma_f$ is the unit weight and $\nu_d$ the dynamic viscosity of the pore fluid. For an isotropic poroelastic medium, the tensors reduce to $\xi = \xi \mathbf{1}$, $k = k \mathbf{1}$ and $k_f = k_f \mathbf{1}$. $F$ is a complex frequency dependent friction factor (Biot [2], Smeluders [18]).
The poroelastic constitutive equations can be summarized as (Biot [2]):

\[
\begin{align*}
\mathbf{t} & = 2\mu^s \varepsilon^s + \lambda^s \text{tr} \varepsilon^s \mathbf{1} + \alpha \pi_p \mathbf{1} \\
\pi_p & = \alpha M \text{tr} \varepsilon^s + M \zeta
\end{align*}
\]

(3)

where \( \mu^s \) and \( \lambda^s \) are the complex Lamé coefficients which account for the hysteretic material damping in the solid skeleton; \( \alpha = 1 - K_d/K_s \) and \( M = (n/\lambda^l + (\alpha - n)/K_s)^{-1} \) are the Biot coefficients. \( K_d, K_s \) and \( \lambda^l \) are the bulk moduli of the drained solid skeleton, the solid matrix and the pore fluid respectively.

The dynamic behaviour of an unsaturated porous medium with a small gas fraction can be described by Biot’s two-phase theory if the influence of the gas phase on the compressibility of the fluid phase is accounted for (Smeulders [18]). Therefore, the fluid bulk modulus \( \lambda^f \) is decomposed as:

\[
\frac{1}{\lambda^f} = \frac{s}{\lambda^l} + \frac{1-s}{\lambda^g}
\]

(4)

where \( s \) denotes the liquid fraction. \( \lambda^l \) is the liquid bulk modulus and \( \lambda^g \) the effective bulk modulus of the gas phase, which relates the averaged bubble volume \( V_g \) to a change in liquid pressure \( p_\infty \) far away from the bubble:

\[
\frac{1}{\lambda^g} = -\frac{1}{V_g} \frac{\partial V_g}{\partial p_\infty}
\]

(5)

A description of wave propagation and attenuation in unsaturated porous media should incorporate the dynamic gas bubble behaviour. Gas bubbles in liquid can vibrate and have a fundamental resonant frequency. Damping of oscillating gas fractions in a porous medium depends upon the shape and structure of the gas fractions. Accounting for the spherical shape of the gas bubble, as revealed by visualisation experiments, equation (5) can be simplified as \( \lambda^g = -R/3 \frac{\partial p_\infty}{\partial R} \) with \( R \) the bubble radius. An expression for \( \lambda^g \) can be found if the fluid pressure \( p_\infty \) far away from the gas bubble is known. In a spherical coordinate system with the origin in the centre of the gas bubble, \( p_\infty \) follows from:

\[
p_\infty = p(R_0) + \int_{R_0}^{\infty} \frac{\partial p}{\partial r} \, dr
\]

(6)

where \( R_0 \) is the bubble radius and \( p(R_0) \) is the fluid pressure at the bubble surface in a fixed reference equilibrium state. For small gas fractions, the medium outside the bubble can be considered as saturated and Biot’s poroelastic theory is used. A rigid solid skeleton is assumed as the bubble motion is considerably larger than the solid skeleton motion. These assumptions allow to derive an expression for the pore pressure gradient, which incorporates the effect of acoustic and Darcy damping. The expression for \( p(R_0) \) introduces thermal and viscous damping. The final expression
for the effective bulk modulus $\lambda^g$ of the gas phase is (Smeulders [18]):

$$
\lambda^g = \left( n_p p_{g0} - \frac{2}{3} \frac{\sigma}{R_0} \right) + \frac{4}{3} \sqrt{\alpha/\omega_c} \nu \omega - \frac{1}{3} \nu \sqrt{\alpha/\omega_c} F
$$

with $n_p$ the polytropic constant, $p_{g0}$ the gas bubble pressure, $\sigma$ the surface tension, $k_{re} = \sqrt{\alpha/C_f \sqrt{\omega_c^2 - i \omega \omega_c F}}$ the effective wave number, $C_f$ the speed of sound in the fluid and $\omega_c = \nu m/k_f \rho_f$ a characteristic frequency. $\lambda^g$ is complex and frequency dependent. The imaginary part causes a phase shift between the change in gas bubble volume and fluid pressure far away from the gas bubble. This gives rise to an energy dissipating mechanism.

3 Spectral formulation

A spectral element formulation for wave propagation in dry and saturated poroelastic media has been presented by Degrande [4] and Degrande and De Roeck [5]. Smeulders’ modification of Biot’s poroelastic theory allows to extend the applicability of this method to unsaturated poroelastic media containing a small amount of air. Due to space limitations, we will only briefly recapitulate some important ideas and refer the reader to [4, 5] for more details.

A decomposition of the displacement vectors in the Navier equations in function of scalar wave potentials allows to decouple the in-plane motion (P-SV) from the out-of-plane motion (SH). Integral transforms are used to obtain a separation of variables in the governing PDE. A Fourier transformation is used from the time ($t$) to the frequency ($\omega$) domain, while a Fourier (in-plane deformation) or a Hankel (axisymmetry) transformation is employed to transform the horizontal or radial coordinate ($x$ or $r$) to the wave number domain ($k_x$ or $k_r$). The PDE are transformed into ODE, for which analytical solutions can be proposed. The dispersion relations reveal the existence of two P-waves (P1 and P2) and a single S-wave that are both dispersive and attenuated and involve a coupled motion of the solid skeleton and the pore fluid (Biot [2]). The response of the medium depends on a dimensionless frequency $\chi$, defined as the ratio of the excitation frequency $\omega$ and the characteristic frequency $\omega_c$. Frozen mixture and a priori-dissipative behaviour are the two extremes for limiting low and high values of $\chi$. Smeulders [18] has demonstrated and measured how a small amount of air influences this behaviour.

The analytical solutions are used as shape functions in a displacement formulation of a saturated (or unsaturated) layer and throw-off element. For a layer element, the incident and reflected kernel wave potentials $\tilde{a}^I$ and $\tilde{a}^R$ are used. A tilde on a variable denotes its representation in the wave number domain. The 6 by 6 complex symmetrical stiffness matrix $\tilde{K}^e$ relates the displacements $\tilde{u}^e$ and tractions $\tilde{T}^e$ in both nodal points of the element.
layer element: \( \mathbf{K}^c \mathbf{u}^c = \mathbf{T}^c \). The (un)saturated throw-off element models the propagation of waves in a semi-infinite halfspace. As the amplitudes of the waves should be non-increasing functions of the distance travelled in the wave propagation direction, only outgoing waves \( \mathbf{a}^f \) are considered. This results in the formulation of 3 by 3 complex symmetrical throw-off element stiffness matrix \( \mathbf{K}^c \). The element matrices \( \mathbf{K}^c \) are assembled in a global complex, symmetrical, banded stiffness matrix \( \mathbf{K}^S \), relating the nodal tractions \( \mathbf{T}^S \) and displacements \( \mathbf{u}^S \):

\[
\mathbf{K}^S \mathbf{u}^S = \mathbf{T}^S
\]  

(8)

This assembly establishes the equilibrium at each horizontal interface, assuming the continuity of the displacements. As analytical solutions are used as shape functions for the formulation of the element stiffness matrices, the mass distribution is treated exactly without the need of subdividing a member into smaller elements. The following applications are envisaged:

**Amplification problems.** The amplification of a plane wave of given type, frequency and angle of incidence in a multilayered medium is calculated. This procedure is used to calculate the free field response in a substructure approach to solve a dynamic soil-structure interaction problem. Another application is the determination of the acoustical impedance of multilayered porous materials.

**Surface waves.** The natural modes of vibration of a multilayered half-space follow from equation (8) where \( \mathbf{T}^S \) is zero. The modes correspond to the non-trivial solutions \( \mathbf{u}^S \). Their existence conditions follow from the transcendental eigenvalue problem: \( \det \mathbf{K}^S = 0 \), which has an infinite number of solutions. It is solved by search techniques as Powell’s hybrid method (Draelants [8]).

**Forced vibrations.** At each node of the interface between two elements, the nodal displacements or tractions can be prescribed. The frequency and spectral content of the temporal and spatial loading distribution is calculated with a FFT algorithm and an analytical evaluation respectively. Equation (8) is solved for each wave number and frequency. The inverse wave number integrals are evaluated numerically. In view of the cost involved with the evaluation of the functions \( \mathbf{u}^S \), the oscillatory behaviour of \( \mathbf{u}^S \) and the kernel of the integral transform and the presence of sharp peaks in \( \mathbf{u}^S \) in the vicinity of surface wave poles, an efficient quadrature scheme is needed. A self-adaptive algorithm is used, based on a modified Filon scheme, concentrating abscissa around regions of sharp variations in \( \mathbf{u}^S \) and taking advantage of previously computed values of \( \mathbf{u}^S \) (Van den Broeck et al. [6]). An inverse FFT is used to return from the frequency to the time domain.
4 Numerical example

We study the influence of a moving ground water table on the propagation of transient waves in an axisymmetric halfspace, consisting of a sand of Mol. The case of (a) a dry halfspace, (b) a saturated halfspace, (c) a halfspace with a water table at a depth $H$ below the free surface and (d) an unsaturated layer of height $H$ on top of a saturated halfspace are considered. The solid skeleton is the same in all cases. (a) $(H = \infty)$ and (b) $(H = 0)$ are limiting cases of case (c) (Philippacopoulos [14, 16]). Results are presented in the wave number-frequency and the space-time domain.

Material characteristics
Sand of Mol is composed of subangular quartz particles with a mean grain diameter $d_{50} = 0.195$ mm; the connected pore space is saturated with water. The porosity $n$ is equal to 0.388, while the sand grain and pore fluid density are equal to $\rho_s = 2650$ kg/m$^3$ and $\rho_f = 1000$ kg/m$^3$. The solid skeleton has a Young’s modulus $E_s = 2.983 \times 10^8$ N/m$^2$ and a Poisson’s ratio $\nu_s = 1/3$. For the calculations in the wave number domain, a transversal and dilatational damping ratio $\beta_s^t = \beta_s^p = 0.005$ are used. These values are sufficiently high to remove the surface wave poles from the real horizontal wave number axis. As the observed material damping in real sandy soils is higher, $\beta_s^t = \beta_s^p = 0.025$ are used for the time domain calculations. The bulk modulus of the pore fluid equals $\lambda = 2.2 \times 10^9$ N/m$^2$. According to the classical soil mechanics’ assumption, the Biot coefficient $\alpha = 1.0$. The Darcy permeability $k = 1.0 \times 10^{-4}$ m/s; a frequency dependent corrector factor $F$ is used to account for the dynamic permeability. The dynamic viscosity of water equals $\nu_d = 1.002 \times 10^{-3}$ Ns/m$^2$. Inertial coupling due to the tortuosity of the pores is accounted for by a tortuosity factor $\alpha = 2.7$. In the unsaturated case, the presence of 0.1 % of gas bubbles with radius $R = 0.001$ m is assumed. The density of the air is equal to $\rho_g = 1.205$ kg/m$^3$, the gas bubble pressure is $p_{g0} = 1.0 \times 10^5$ N/m$^2$ and the surface tension is $\sigma = 0.07$ N/m.

Results in the wave number domain
The halfspaces are loaded by a unit vertical force $T_z(r, z = 0, t) = \delta(r)\delta(t)$ N/m on the solid skeleton so that the results are not influenced by the spectral content of the loading. Figure 1 shows $\text{Re}[\hat{u}_z^s(k, z = 0, \omega)]$, the real part of the free surface solid skeleton vertical displacement, in function of $\omega$ and the dimensionless horizontal wave number $\tilde{k}_r = k_r C_s / \omega = C_s / C$, with $C_s$ the S-wave velocity in the drained solid skeleton and $C$ the phase velocity. The frequency varies between 10 and 100 Hz and $\tilde{k}_r$ between 0 and 2.

Dry halfspace (figure 1(a)). In the 1D case ($\tilde{k}_r = 0, C = \infty$), the P- and S-wave propagation are uncoupled. For vertical loading, only P-waves are generated. For $0 \leq \tilde{k}_r \leq s$ ($\infty \geq C \geq C_p$), both P and S waves are
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propagating. The coefficient \( s = C_s/C_p \) only depends on Poisson’s ratio and is equal to 0.5 in this case. For \( s \leq \tilde{k}_r \leq 1 \) \((C_p \geq C \geq C_s)\), the S-wave is propagative, while the P-wave is inhomogeneous with exponential decay in the \( z \)-direction. For \( \tilde{k}_r > 1 \) \((C_s \geq C)\), both waves are inhomogeneous. A peak corresponding to the Rayleigh surface wave appears. This wave has a phase velocity \( C = C_r \) that satisfies the classical Rayleigh wave equation. Due to the introduction of hysteretic material damping in the solid skeleton, the root of the secular equation is complex, i.e. the throw-off element stiffness matrix is never singular in the real wave number domain.

**Saturated halfspace** (figure 1(b)). The frequency range in figure 1 corresponds to low values of \( \chi \). The response of the medium tends to the response of a frozen mixture or an equivalent undrained medium. This explains the higher P-wave velocity; the S-wave velocity is only weakly affected as it is only influenced by the change in density. The complex secular equation which governs the surface wave behaviour has been studied by Jones [10] and Deresiewicz [7]. The surface wave velocity is lower than in the dry halfspace, while the lower compressibility results in smaller displacements. In the present case, the attenuation is mainly due to the hysteretic material damping in the solid skeleton. Additional attenuation can be attributed to relative motion between both phases, as some drainage may occur at the free surface.

**Dry layer on a saturated halfspace** (figure 1(c)). It is convenient to introduce a dimensionless depth \( \tilde{H} = H\omega/C_s = 2\pi H/\lambda_s \) which relates the depth \( H \) of the water table to the wavelength \( \lambda_s \) of the S-wave in the layer. \( \tilde{H} \) can also be interpreted as the ratio of the excitation frequency \( \omega \) to the first eigenfrequency of the layer built in at its base. The latter is equal to \( \omega_p = 2\pi C_p/4H \) for vertical excitation, i.e. \( \tilde{H} = (\pi/2s)(\omega/\omega_p) \). In figure 1(c), \( \tilde{H} \) varies between 0 and 11.95, including the first \((\tilde{H} = \pi)\) and second \((\tilde{H} = 3\pi)\) resonant frequency. For small values of \( \chi \), the presence of a ground water table introduces layering. For limiting low and high values of \( \tilde{H} \), the medium behaves as a saturated or a dry halfspace respectively. This is apparent regarding the contribution of the dispersive fundamental surface wave. At \( \tilde{H} = \pi \) and \( \tilde{H} = 3\pi \) higher dispersive modes appear, which will influence the response of the medium.

**Unsaturated layer on a saturated halfspace** (figure 1(d)). The influence of the layering, introduced by the presence of a small amount of air up to a depth \( \tilde{H} \), depends on a dimensionless frequency \( \tilde{H} \), defined as a ratio of the frequency and the first eigenfrequency of the unsaturated layer built in at its base. For low and high values of \( \tilde{H} \), the medium behaves as a saturated and unsaturated halfspace respectively. Higher modes appear at the resonant frequencies of the unsaturated layer built in at its base. As the
small amount of air contributes to a higher attenuation, the contribution of these modes will be limited, especially for higher frequencies.

![Figure 1: Real part of the solid skeleton displacement at the free surface of a poroelastic halfspace in function of $k_r$ and $\omega$: (a) dry and (b) saturated halfspace; (c) dry and (d) unsaturated layer on a saturated halfspace.](image)

**Results in the time domain**

The transient response of the halfspaces due to unit distributed vertical traction on an area with radius $R = 0.5\,\text{m}$ is calculated. The transient loading function is half a period of a squared sine function with period $T = 0.06\,\text{s}$. A time delay $t_d = 0.02\,\text{s}$ is applied. The response in the frequency domain is obtained by an inverse wave number transformation from the wave number domain to the domain of the radial coordinate $r$. An
adaptive generalized Filon quadrature is used, with a target integration interval $\Delta k_r = 0.001$ and a maximum wave number $k_r^{\text{max}} = 32.0$. A FFT algorithm is used to return from the frequency to the time domain. The sampling interval is $\Delta t = 0.001\,\text{s}$, and the assumed period of the signals is $T = 0.512\,\text{s}$, corresponding to $2^9$ sampling points. Figures 2(a), 2(b) and 2(c) show the frequency content and time history of the vertical acceleration in the solid skeleton on the free surface at receiver distances $r (i = 1, 10$ and $r = 5.0\,\text{m})$. The following comments follow immediately from the discussion on the wave number-frequency domain results.

**Dry halfspace.** The response in the dry poroelastic halfspace is used as a reference and represented as a dashed line in all figures. The P-wave is faster than the S-wave, which is immediately followed by the surface wave, having the most important contribution to the time signal. The observed attenuation is due to geometric and material damping and primarily affects the high frequency components.

**Saturated halfspace** (figure 2(a)). The time histories show a dispersive main pulse according to the surface wave, which travels with a smaller velocity than the surface wave in the dry halfspace. The pulse according to the P1-wave arrives earlier than the P-wave in the dry halfspace. The amplitudes of the surface wave are lower as the compressibility of the saturated medium is lower than the compressibility of the dry medium. As the dimensionless frequencies $\chi$ are rather low, the attenuation due to the viscous coupling between both phases is low. Attenuation is mainly due to geometric and material damping in the solid skeleton.

**Dry layer on a saturated halfspace** (figure 2(b)). For small values of the dimensionless frequency $\chi$, the water table introduces layering. This influences the propagation of the fundamental surface mode and gives rise to higher surface modes. Close to the source, the direct P-wave is the fastest wave, while for higher source-receiver distances, a refracted P-wave arrives first. The latter originates from critical incidence of the P-wave on the interface between the layer and the halfspace. Resonance at the first resonance frequency of the dry layer built in at its base is observed.

**Unsaturated layer on a saturated halfspace** (figure 2(c)). The response of this case is situated between the limiting cases of a dry layer on a saturated halfspace and a saturated halfspace. The surface wave is similar as the surface wave in the saturated halfspace. The presence of a small amount of air in the pores of the top layer only affects the compressibility. More pronounced but still small differences can be observed regarding P-wave propagation. A weak layering effect, due to a change in material behaviour at the interface of both materials can be observed.
Figure 2: Comparison of the frequency content (LHS) and time history (RHS) of the vertical acceleration in the solid skeleton at the free surface of a dry halfspace (dashed line) and, in solid line: (a) saturated halfspace; (b) dry and (c) unsaturated layer on a saturated halfspace.
5 Conclusion

A unified spectral element approach has been presented to study transient wave propagation in dry, saturated and unsaturated poroelastic layered media. Therefore, Biot’s poroelastic theory has been extended to cover the unsaturated case, where a small amount of air is present in the pores. The formulation enables the solution of a wide range of problems in geophysical and civil engineering, as demonstrated in a numerical example.

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