Efficiency of absorbing boundary conditions for fluid-saturated porous media
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Abstract
For the dynamic analysis of fluid-saturated porous media, general expressions of the absorbing boundary conditions in time domain are presented in terms of $u-w$ formulation, based on Biot's two phase mixture theory and 0th-order paraxial approximation. Numerical computation is conducted by using a 2D FE-program with the absorbing boundary conditions which is developed for dynamic effective stress liquefaction analysis. The results for a 1D linear model show the validity and efficiency of proposed absorbing boundary conditions. Also applicability of the absorbing boundary conditions to nonlinear problems is illustrated for a 2D liquefaction problem.

1 Introduction
The analysis of dynamic phenomena in fluid-saturated porous media is of great interest in geotechnical and earthquake engineering. Biot[1] first established the framework in the formulation of the dynamic response of fluid-saturated elastic-porous media. The dynamic analysis is usually implemented via numerical methods involving discretization of both spatial and temporal domains. A variety of approaches by finite element models has been developed for such problem. These approaches can account for the complex geometry, inhomogeneity and nonlinear behavior. However, compared with the finite element scheme applied to the single-phase media, the computation effort increases substantially due to the additional degrees of freedom associated with the pore fluid. In order to reduce the cost of analysis, the computational model is restricted to a finite domain. Consequently, it is necessary to devise special boundary techniques to incorporate the radiation condition of the truncated unbounded domain into the
finite computational model.

In the dynamic analysis of fluid-saturated porous media, Modaressi and Benzenati [2,3] proposed an absorbing boundary element for the \( u-p \) formulation and presented an illustrative example of the efficiency of the proposed boundary. For \( u-w \) and \( u-U \) formulations, however, no absorbing boundary may be available. Therefore, this study is devoted to this issue.

2 Governing equation

The porous medium can be viewed as a mixture composed of solid and pore fluid phases. The dynamic equilibrium equation for the solid-fluid phase and the generalized Darcy law for the dynamic equilibrium of the pore fluid can be written as [1,4]

\[
L^T \sigma + \rho b = \rho \ddot{u} + \rho_f \ddot{w} \quad (1)
\]
\[
-\nabla p + \rho f b = \rho \ddot{u} + \frac{\partial f}{\partial \dot{w}} + \frac{\rho \rho_f}{k} \ddot{w} \quad (2)
\]

where a superposed dot indicates a time derivative and a vector matrix notation is used to represent tensors; i.e. \( \sigma^T = (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}) \); \( u^T = (u_x, u_y, u_z) \); \( w^T = (w_x, w_y, w_z) \); \( b^T = (b_x, b_y, b_z) \); \( \nabla^T = (\partial / \partial x, \partial / \partial y, \partial / \partial z) \) and

\[
L^T = \begin{pmatrix}
\frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\
0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & 0 \\
0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{pmatrix}
\]

Here, \( u \) and \( w \) are the solid phase displacement and the relative pore fluid displacement, respectively; \( \sigma \) is the total stress; \( b \) is the body force; \( p \) is the pore fluid pressure; \( n \) is the porosity; \( \rho \) and \( \rho_f \) are the density of the bulk solid-fluid mixture and the density of the pore fluid, respectively, so that \( \rho = (1-n)\rho_s + n\rho_f \), where \( \rho_s \) is the density of the solid grain; \( k \) is the isotropic permeability coefficient.

The stress-strain relationship of a linear, isotropic, elastic material can be written as

\[
\sigma = D \varepsilon - \alpha Q \varepsilon_p
\]
\[
p = -\alpha Q m^T \varepsilon - Q \varepsilon_z
\]

where \( \varepsilon = Lu \) and \( \varepsilon_z = \nabla^T w \) are the strain in the solid and the volumetric strain in the pore fluid, respectively; \( D \) is the drained material stiffness matrix; \( m^T = (1, 1, 1, 0, 0, 0) \) is equivalent to the Kronecker's delta; \( \alpha \) and \( Q \) are related with materials through

\[
\alpha = 1 - K_d K_s \quad \text{and} \quad 1/Q = n/K_f + (\alpha - n)/K_s
\]

where \( K_s \) and \( K_f \) are the bulk moduli of the solid and pore fluid, respectively; \( K_d \)
is the bulk modulus of the solid skeleton.

3 Absorbing boundary conditions

We now consider the infinite domain as shown in Figure 1. Firstly, we assume that (1) there is continuity of the stress, displacement, pore fluid pressure and pore fluid flow on the interface boundary; (2) the media are linear, isotropic and homogeneous at the vicinity of the interface boundary.

After zeroth-order paraxial approximation [5] of Biot's dynamic equation in frequency domain, an absorbing boundary condition in time domain is obtained as mixed boundary condition by taking the inverse Fourier transform. Then the absorbing boundary condition in \( u-w \) formulation establishes a relation between stresses and displacements as impedance;

\[
\begin{bmatrix}
\tau_{zx}, \tau_{zy}, \tau_{zz}, 0, 0, -p
\end{bmatrix}^T = - \begin{bmatrix}
A_{uu} & A_{uw} \\
A_{uw}^T & A_{ww}
\end{bmatrix} \begin{bmatrix}
\ddot{u} \\
\ddot{w}
\end{bmatrix}
\]  

(7)

with

\[
A_{uu} = \begin{pmatrix}
\rho V_3 & 0 & 0 \\
0 & \rho V_3 & 0 \\
0 & 0 & \rho V_1
\end{pmatrix};
A_{uw} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \alpha Q/V_1
\end{pmatrix}
\]  

(8)

\[
A_{ww} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & Q/V_1
\end{pmatrix}
\]

(9)

where \( V_1 \) and \( V_3 \) are velocities of the first dilatational wave and shear wave, respectively. For the linear material, \( V_1 \) and \( V_3 \) are given by

\[
V_1 = \sqrt{(\lambda + 2G + \alpha^2 Q)/\rho};
V_3 = \sqrt{G/\rho}
\]

(10)

where \( \lambda \) and \( G \) are the Lame's constants. From the above absorbing boundary conditions, it can be seen that the zeroth-order paraxial boundaries correspond to consistent dampers. Therefore, it can be noted that these absorbing boundaries are also equivalent to the viscous boundaries.

By using the projection matrix \( P \) for equation (7) and considering the motion of the free field during earthquakes, the absorbing boundary condition in the global Cartesian coordinate system can be given by

\[
\begin{bmatrix}
\dddot{t} \\
\dddot{m}
\end{bmatrix} = \begin{bmatrix}
\dddot{u} \\
\dddot{w}
\end{bmatrix} = \begin{bmatrix}
P^T A_{uu} P & P^T A_{uw} P \\
P^T A_{uw}^T P & P^T A_{ww} P
\end{bmatrix} \begin{bmatrix}
\dddot{u} - \dddot{w} \\
\dddot{w} - \dddot{w}
\end{bmatrix}
\]

(11)

where a superscript \( f \) at a variable represents the contribution from the motion of the free field.

4 Numerical examples

The proposed method is incorporated into the finite element computer program NUW2 (nonlinear \( u-w \) analysis in two-dimensions). In the calculations, the
four-node bilinear isoparametric element is used. The Newmark parameters in the integration scheme are chosen as $\gamma = 0.60$ and $\beta = 0.3025$ in order not to obscure the interpretation of the results.

### 4.1 Validity and efficiency in 1D linear model

The load boundary conditions are a unit step stress and free fluid flow at the top surface of media. The material properties used for the numerical analyses are the same as ones in Ref. 6 which was used for an analytical solution [6] of one-dimensional fluid-saturated porous media as shown in Figure 2; $\lambda = 0.8333$ kPa; $\mu = 1.25$ kPa; $K_v = \infty$; $K_f = 39.999$ kPa; $\rho = 0.306$ kg/m$^3$; $\rho_f = 0.2977$ kg/m$^3$; $k_f = 0.01425$ m/s; $n = 0.333$; $\alpha = 1.0$. In order to judge the efficiency and effectiveness of the absorbing boundary, either the absorbing boundary or the fixed boundary is applied to the artificially truncated finite boundary at a depth of 100 m.

Figures 3 and 4 show time histories of pore fluid pressures and solid displacements, respectively, at the depths of 5 m and 45 m for the transient wave propagation problem. It can be observed from the related results that there is a good comparison between the analytical solution and the numerical one obtained by the finite element method with the absorbing boundary. The displacement is considerably underestimated in the case of using the artificially fixed boundary. Although a small numerical oscillation in the pore fluid pressure exists in the case of using the absorbing boundary, it decreases quickly as time goes on. Thus, it is concluded that the use of the proposed absorbing boundary is an efficient and effective way to model the far field of the system for the transient wave problem in one-dimensional infinite media.

### 4.2 Applicability to 2D nonlinear model

Next, we consider the applicability of the absorbing boundary conditions to 2D nonlinear media. Saturated soil deposits which are divided into two layers and resting on the rigid base are modelled to 70 finite element meshes (Model 1) and 60 ones (Model 2), and subjected to an in-plane El Centro earthquake (1940 NS) with the maximum acceleration of 0.2g at the rigid base as shown in Figure 5. In this case, real time nonlinear seismic motions of 2D shear columns which are equivalent to free field motion are used as an efficiency measure of the absorbing boundary conditions, comparing with the motions of the lateral boundaries of 2D site models.

We assume that the soil deposits are saturated, of nonlinear plane strain, and drained at the top surface of the medium. The material properties are chosen as; $\nu = 0.33$, $n = 0.4$, $\rho = 2.0$ t/m$^3$, $\rho_f = 1.0$ t/m$^3$, $k = 1.0 \times 10^{-5}$ m/s, $\alpha = 1.0$, $K_v = 2.0 \times 10^6$ kPa. The liquefaction parameters defined in the strain space multimechanism model are given in Table 1.

Figures 6 and 7 compare the scale effects of two soil deposit models having proposed absorbing (viscous) boundary condition. Scale effect of soil deposit models is more sensitive to pore water pressure than displacement.
Absorbing boundary condition becomes more efficient for expanded model (:Model 1) than small model (:Model 2).

To illustrate the efficiency of proposed absorbing boundary conditions, Figure 8 for free boundaries is compared with Figure 7. Free boundary conditions at the lateral side boundaries give considerable errors to the seismic responses of boundary meshes, compared with the absorbing boundary conditions.

5 Conclusion

Based on Biot's two-phase mixture theory and the paraxial approximation, the absorbing boundary condition for \( u-w \) formulation in the time domain is presented for the dynamic analysis of infinite fluid-saturated porous media. Numerical results show that the proposed absorbing boundary is of high ability to absorb the energy of the reflected waves.

Through the examples involving the transient dynamic problems in the infinite fluid-saturated porous media in one and two dimensions, the accuracy and efficiency of using the proposed absorbing boundary have been verified. It is concluded that, compared with usually truncated boundary for simulating the effect of the far field of the system, the use of the proposed method can provide very accurate results for the solution of linear and nonlinear transient dynamic problems in infinite fluid-saturated porous media because the spuriously reflected waves in the system can be greatly supressed.

References

Figure 1 Typical infinite media

Figure 2 1D transient dynamic problem

Figure 3 Fluid pore pressures in 1D transient dynamic problem

Figure 4 Displacements in 1D transient dynamic problem
Figure 5  Finite element models of nonlinear fluid-saturated soil deposits

Table 1  Soil parameters for liquefaction analysis

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Upper layer</th>
<th>Lower layer</th>
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<tr>
<td>$Gm_0 (kPa)$</td>
<td>22990</td>
<td>65030</td>
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<td>$p_1$</td>
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<td>0.50</td>
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<td>$p_2$</td>
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<td>$w_1$</td>
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<td>$s_1$</td>
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<td>0.005</td>
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<td>$c_1$</td>
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<td>1.70</td>
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<td>$\phi'_f$</td>
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<td>37°</td>
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<tr>
<td>$\phi'_p$</td>
<td>28°</td>
<td>28°</td>
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<tr>
<td>$H_m$</td>
<td>0.30</td>
<td>0.30</td>
</tr>
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Figure 6  Response of point B in Model 2 with absorbing boundary

Figure 7  Response of point A in Model 1 with absorbing boundary

Figure 8  Response of point A in Model 1 with free boundary