Effect of channel slope on flow characteristics of undular hydraulic jumps

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Abstract

Undular hydraulic jumps in sloping rectangular channels have been experimentally investigated. The upper limit of the inflow Froude number for the formation of an undular jump depends on the channel slope; if the channel slope is steeper than the critical slope. The streamwise changes of the surface profile, the bed pressure head, and the bed velocity in undular jumps have been characterized by the channel slope. The effects of channel slope on the wave height, the wave length, and the amplitude of undular jumps have been shown. Also, surface profiles of undular jumps have been compared with theoretical results proposed by Grillhofer and Schneider. The applicability of the theoretical results is discussed.

Keywords: undular jump, hydraulic jump, open channel flow, channel slope.

1 Introduction

The undular jump is a well-known transitional phenomenon from supercritical to subcritical flows with undulations of water-surface. The undular jump often occurs at the downstream of low drop structures or in a transitional region from steep to mild sloping channels.

Ohtsu et al. [1-3] have shown that the flow conditions of undular jumps in horizontal rectangular channels are characterized by the inflow Froude number, the thickness of the boundary-layer at the toe of jump, aspect ratio, and Reynolds number. Also, Ohtsu et al. [1-3] have classified the flow conditions of undular jumps, and have clarified the hydraulic conditions for undular jump formations.

But, there are few information for the effect of channel slope on the undular jump formation. For design purpose, it is necessary to know the effect of channel slope on characteristics of undular jumps.
In this paper, characteristics of undular jumps in sloping rectangular channels have been presented under a wide range of experimental condition. The flow conditions of undular jumps in sloping channels have been clarified. The upper limit of the inflow Froude number for the formation of undular jumps depends on the channel slope if the channel slope is steeper than critical slope. Characteristics of undular wave (the wave length, the wave height, and the wave amplitude) have been shown. For water-surface profile of undular jump in steep sloping channels, the applicability of the theoretical result of Grillhofer and Schneider [4] has been discussed.

2 Experiments

The experiments were performed in smooth rectangular channels under the experimental conditions as shown in Table 1. The channel slope can be adjusted by using a geared mechanism. In order to measure the water-surface accurately, the point gauge and the water gauge with a servomechanism were used (Ohtsu and Yasuda [5]). The location of undular jumps was controlled by the upstream sluice gate and the downstream sharp edged weir (Ohtsu et al. [3]).

### Table 1: Experimental condition.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Channel slope : $\tan \Theta$</td>
<td>0 to 1/80</td>
</tr>
<tr>
<td>Channel width : $B$ (cm)</td>
<td>80 to 165</td>
</tr>
<tr>
<td>Supercritical flow depth : $h_1$ (cm)</td>
<td>4.7 to 13.9</td>
</tr>
<tr>
<td>Discharge : $Q$ (m$^3$/s)</td>
<td>0.0575 to 0.128</td>
</tr>
<tr>
<td>Reynolds number : $Re$</td>
<td>$6.5 \times 10^4$ to $1.8 \times 10^5$</td>
</tr>
<tr>
<td>Inflow Froude number : $F_1$</td>
<td>1.07 to 3.16</td>
</tr>
<tr>
<td>Aspect ratio : $B/h_1$</td>
<td>2.9 to 37.4</td>
</tr>
</tbody>
</table>

If the value of the inflow Froude number $F_1$ is larger than 1.2, lateral shock waves are formed near the toe of undular jump (Fig.1) (Ohtsu et al. [1-3]) $(F_1 = v_1/(gh_1)^{1/2}$, $v_1$=average velocity at the toe of undular jump, $h_1$=supercritical depth at the toe of undular jump, $g$=acceleration due to gravity). The occurrence of lateral shock waves may be explained as follows (Ohtsu et al. [3]): The adverse pressure gradient arises from the existence of the subcritical flow depth in the jump, and the degree of a side-wall-boundary-layer development is extended from the toe of jump. Then, the surface-flow velocity near the side walls is apt to become critical, such that the lateral shock waves form from both sides of the toe section.

If the lateral shock waves cross at the upstream of the first wave crest [for a narrow channel (Ohtsu et al. [3])] [Fig.1 (a)], the undular wave becomes three-dimensional. In this case, the effect of aspect ratio on the flow condition of undular jumps is not negligible. While, if the lateral shock waves do not cross at
the upstream of the first wave crest [for a wide channel (Ohtsu et al. [3])] [Fig. 1(b)], the undular wave formation except near the side walls is almost two-dimensional.

The experiments were made under the condition in which the lateral shock waves do not cross at the upstream of the first wave crest [Fig. 1(b)]. Accordingly, the effect of aspect ratio on flow characteristics of undular jumps is negligible.

The experiments were also performed under the range of \( Re \geq 6.5 \times 10^4 \) (\( Re = v_l h_1 / v; \ v = \) kinematic viscosity). In this range, the effect of Reynolds number on flow characteristics of undular jumps is negligible, and the Froude similarity law is satisfied (Ohtsu et al. [1-3]).

In the experiments, the undular jumps were formed in order that a turbulent boundary layer at the toe of undular jump was fully developed. Fig. 2 shows an example of velocity profile at the toe of undular jump. The velocity profile is approximated as the one-seventh-power law (Fig. 2). Here, \( y = \) length normal to the channel bed, \( u = \) mean velocity at \( y = y' \), and \( U_{\text{max}} = \) maximum velocity.

![Lateral shock waves](image)

Flow

(a) Undular jump in narrow channel
(The lateral shock waves cross at the upstream of first wave crest.)

Flow

(b) Undular jump in wide channel
(The lateral shock waves do not cross at the upstream of first wave crest.)

Figure 1: Formation of lateral shock waves.

### 3 Flow conditions of undular jumps and hydraulic conditions for undular jump formations

The flow conditions of undular jumps in sloping channels are explained as follows:

\( 1 < F_l \leq 1.2 \) : The water surface has small undulations, and two-dimensional undulations are formed [Fig.3 (a)].
1.2 < \( F_1 \leq F_{\text{limit}} \): Nonbreaking stable undulations continue far downstream along the center part of the channel [Fig. 3 (b)]. Here, \( F_{\text{limit}} \) shows the upper limit of the inflow Froude number for the formation of nonbreaking undular jumps.

\( F_1 > F_{\text{limit}} \): The first wave always breaks at the central part of the channel with stable undulations continuing downstream [Fig. 3 (c)]. If the value of \( F_1 \) further increases, at a certain stage (\( F_1 > F_{1u} \)) [Fig. 3 (d)], a weak classical jump is formed and stable undulations are not observed.

The above mentioned flow conditions in sloping channels are similar to those in horizontal channels.

![Flow](image)

**Figure 2:** Example of velocity profile at toe of undular jump.

(a) Two dimensional nonbreaking undular jump (\( 1 < F_1 \leq 1.2 \))

(b) Nonbreaking undular jump (\( 1.2 < F_1 \leq F_{\text{limit}} \))

(c) Breaking undular jump (\( F_{\text{limit}} < F_1 < F_{1u} \))

(d) Classical jump (\( F_1 \geq F_{1u} \))

**Figure 3:** Flow conditions.
Fig 4 shows the hydraulic conditions for the nonbreaking undular jump formations. The solid line shows the upper limit of the inflow Froude number \( F_{\text{limit}} \) for the formation of nonbreaking undular jumps. If the channel slope becomes smaller than the critical slope, the value of \( F_{\text{limit}} \) is constant value \( F_{\text{limit}} \approx 1.78 \) (Ohtsu et al. [1]). If the channel slope becomes larger than the critical slope, the value of \( F_{\text{limit}} \) depends on channel slope and the value of \( F_{\text{limit}} \) decreases as the channel slope increases.

![Figure 4: Hydraulic conditions for undular jump formations.](image)

4  Effect of channel slope on characteristics of undular jumps

4.1 Streamwise changes of water surface, bed pressure head, and bed velocity

Figs. 5 [(a), (b), and (c)] shows streamwise changes of the surface profile, the bed pressure head, and the bed velocity of undular jumps. In Figs. 5 [(a), (b), and (c)], \( h \)=flow depth at \( x=x \), \( p_b \)=bed pressure, \( \rho \)=density of water, \( v_b \)=bed velocity, \( x \)=streamwise length from the toe of lateral shock wave. Here, the bed pressure and the bed velocity were measured at \( y=1 \text{mm} \).

As shown in Figs. 5 [(a) and (b)], for the cases of horizontal and mild sloping channels, the bed pressure head and the bed velocity change alternately, which corresponds to the profile of undular surface. Also, the averaged flow depth along the channel shows almost constant. For the case of steep sloping channel [Fig. 5 (c)], the bed pressure head and the averaged flow depth along the channel increase with streamwise distance.

4.2 Characteristics of undular wave

If the experimental data of the first wave length \( L_1 \), the first wave height \( h_{m1} \), and the first wave amplitude \( a_1 \) are arranged in accordance with the relation of (1), Figs. 6[(a), (b), and (c)] are obtained.
Figure 5: Streamwise changes of surface profile, bed pressure head, and bed velocity along with streamwise distance.

\[
\frac{L_1}{h_2} \cdot \frac{h_m}{h_1} \cdot \frac{a_1}{h_1} = f(F_1, \Theta)
\]  

where, \( h_2 \) is the sequent depth of the jump in horizontal channel and is calculated by

\[
h_2 = h_1 \left( 8F_1^2 + 1 \right)^{0.5} - 1 / 2.
\]

As shown in Fig. 6 (a), the first wave length \( L_1/h_2 \) for steep sloping channel is shorter than that for mild sloping and horizontal channels. Also, for steep sloping channel, as shown in Fig. 5 (c), the wave length decreases along with streamwise distance. While, for both mild sloping and horizontal channels, the wave length does not change in the region of stable undulations [Figs. 5 (a) and (b)].

As shown in Figs. 6 [(b) and (c)], the first wave height \( h_m/h_1 \) and the first wave amplitude \( a_1/h_1 \) are independent of channel slope. Also, the wave amplitude decreases with streamwise distance [Fig. 5 [(a), (b), and (c)].

In addition, \( L_1/h_2 \) is approximated by equation (2) for mild sloping channel. \( h_m/h_1 \) and \( a_1/h_1 \) are approximated by equations (3) and (4) for horizontal, mild sloping, and steep sloping channels.
4.3 Theoretical approaches on water surface profile of undular jumps

Iwasa [6] derived the following differential equation (5) for predicting the surface profile of undular jumps on the basis of the continuity and the momentum equations by considering vertical acceleration.

\[
hV^2 + \frac{1}{2} gh^2 \cos \Theta + \frac{1}{3} h^2V^2 \frac{d^2 h}{dx^2} - \frac{1}{3} hV^2 \left( \frac{dh}{dx} \right)^2 = 0
\]  

Solving equation (5) gives equations (6) and (7).

\[
\frac{L_i}{h_i} = \frac{0.8}{F_i} + 2.9
\]

\[
\frac{h_{m1}}{h_i} = 1.51F_i - 0.35
\]

\[
\frac{a_i}{h_i} = -1.3F_i^2 + 4.65F_i - 3.35
\]  

Figure 6: Characteristics of first wave.
Here, the equations (6) and (7) are expressed as a solitary wave and a cnoidal wave, respectively. Iwasa showed the surface profile of undular jump by connecting the solitary wave [equation (6)] to the cnoidal wave [equation (7)] at the first wave crest.

Andersen [7] developed the theory on the surface profile of undular jump on the basis of Boussinesq energy equation [equation (8)]. Equation (9) is transformed from equation (8), and the surface profile of undular jump is calculated from equation (9).

The specific energy head
\[ E = h + \frac{v^2}{2g} + \frac{v^2}{3g} \left( h \frac{d^2h}{dx^2} \right) \]  
(8)

where, \( E \)=specific energy head, \( h \)=flow depth at \( x=x \), \( v \)=average velocity at \( x=x \), and \( h_i \)=specific energy head at \( x=x \).

The specific energy head \( E \) can be derived from the Eulerian equation of motion and the continuity equation as follows:
\[ E = h + \frac{v^2}{2g} + \frac{v^2}{3g} \left( h \frac{d^2h}{dx^2} \right) - \frac{v^2}{6g} \left( \frac{dh}{dx} \right)^2 \]  
(10)

Thus, it is understood that equation (8) is an approximated equation obtained by neglecting the forth term of equation (10).

Regarding the first wave height and the first wave length, the theoretical results proposed by Iwasa and Andersen were compared with the experimental results (Ohtsu et al. [1-3]). The authors showed that the theoretical results do not agree with the experimental results under the condition \( (F_1 > 1.2) \) in which the lateral shock wave is formed. Thus, the applicable range of the theories of Iwasa [6] and Andersen [7] is limited.

Recently, Grillhofer and Schneider [4] have developed a new theoretical approach on the water surface profile of undular jump in steep sloping channels. They have derived nonlinear third-order ordinary differential equation (11) by using a perturbation method.
\[ \frac{d^3H_i}{dx^3} - (1 - H_i) \frac{dH_i}{dx} - \beta H_i = 0 \quad (\beta = \frac{\Theta}{3} e^{-3\varepsilon}, F_i = 1 + \frac{3}{2} \varepsilon) \]  
(11)

where, \( H_i \)=non-dimensional perturbation of shape of the free surface at \( X=X \) [\( h=h_i(1+\varepsilon H_i) \)], \( X \)=non-dimensional length along channel bed (\( X=\delta x/h_i \), \( \delta=3\varepsilon^{1/2} \)), and \( \varepsilon \)=perturbation parameter.

Fig. 7 shows the comparisons between theoretical and experimental results for the first wave height and the first wave length. The solid line in Fig.7 shows the theoretical results derived from equation (11). The experimental values are obtained under the condition in which the uniform flow is established at the toe of undular jump. The values of \( F_i \) for a uniform flow is determined under a given
channel slope. In order to calculate the equation (11), the experimentally obtained values of $F_1$ and $\Theta$ are substituted into equation (11). As shown in Fig. 7, a good agreement between theoretical and experimental results for the first wave height $h_{m1}/h_1$ has been obtained under the condition ($F_1 < F_{1\text{limit}}$) in which the nonbreaking undular jump is formed. While, the theoretical values on the first wave length $L_1/h_2$ agree with the experimental results for $F_1 \leq 1.2$ in which lateral shock waves are not formed.

Figure 7: Comparison of theoretical with experimental results for first wave height and first wave length.

Fig. 8 shows surface profiles of undular jumps for both cases of $F_1 \leq 1.2$ and $F_1 > 1.2$. Here, $x=$streamwise length from the first wave crest, and $z=$transverse length from center of channel. As shown in Fig. 8(a), for $F_1 \leq 1.2$, the theoretical results of surface profile agree with the experimental results. While, for $F_1 > 1.2$, the theoretical results of surface profile differ from the experimental results [Fig. 8(b)]. It is considered that the difference between theoretical and experimental results may be caused by the formation of lateral shock waves.

5 Conclusion

The effect of channel slope on flow characteristics of undular jumps in sloping channels has been investigated. The results are summarized as follow:

1. The flow conditions of undular jumps in sloping channels have been classified as Figs. 3 and 4.

2. If the channel slope becomes larger than critical slope, the value of the upper limit of the inflow Froude number for the formation of nonbreaking undular jumps $F_{1\text{limit}}$ decreases as the channel slope increases (Fig. 4). For mild sloping channel, the value of $F_{1\text{limit}}$ is independent of channel slope and is constant value ($F_{1\text{limit}} \approx 1.78$).
(3) For streamwise changes of the flow depth, the bed velocity, and the bed pressure head of undular jump, the difference between steep sloping and mild sloping channels can be shown [Figs. 5 [(a), (b), and (c)].]

(4) The first wave length $L_1/h_2$ for steep sloping channel is shorter than that for mild sloping and horizontal channels [Fig. 6 (a)]. Also, for steep sloping channel, the wave length decreases with streamwise distance [Fig. 5(c)]. While, for horizontal and mild sloping channels, the wave length does not change along the channel [Figs. 5 [(a) and (b)].

The first wave height $h_{m1}/h_1$ and the first wave amplitude $a_1/h_1$ are independent of channel slope [Figs. 6 [(b) and (c)]. Also, the wave amplitude decreases with streamwise distance [Figs. 5 [(a), (b), and (c)].]

(5) The theoretical results on the surface profile proposed by Grillhofer and Schneider [4] has been compared with the experimental results. A good agreement between theoretical and experimental results for the wave height $h_{m1}/h_1$ has been obtained under the condition of $F_1 \leq F_{1\text{limit}}$ in which the nonbreaking undular jump is formed [Fig. 7 (a)]. Also, the theoretical values on the first wave length $L_1/h_2$ agree with the experimental results for the range of $F_1 \leq 1.2$ in which lateral shock waves are not formed [Fig. 7 (b)].

References


