Characteristic value determination for arbitrary distribution

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Abstract

This paper deals with the characteristic value determination from relatively small samples. When using samples the parameters of distribution can only be estimated and the correct characteristic value is unknown. The methods of estimation of characteristic values for several distributions and previously prescribed confidence intervals are presented in this paper. Results are confirmed by simulations.

1 Introduction

In engineering practice the random variables are usually represented by their characteristic values. This approach is suitable for further analysis and design because we can use the fixed values without employing any probability methods. However, when only relatively small sample is available, the characteristic value is only estimated from that sample. The estimate is based on the assumption that the distribution of the variable is known and that its parameters are approximated from a sample. If we review the European standards [3] different distributions are usually prescribed for the determination of the resistance of different materials and for the determination of the resistance of structures: normal, lognormal, Gumbel, etc. For most cases formulae for the 75% confidence interval for the estimates of 5% characteristic values are prescribed. All the standards and the substandards propose the results in the form of tables of coefficients; without any analytical alternatives proposed. We in contrast analyze the problem in rather more general form. Our approach offers the confidence interval for the arbitrary characteristic value, which can be obtained analytically for some distributions. By the use of analytical formulae the tables, similar to those in European standards, are presented for the normal and lognormal distribution. For an arbitrary distribution an approximation
based on analytical equations is developed. The results are verified by the use of simulations. For the normal and lognormal distribution we confirm the analytically developed values. Because of some additional approximations made for an arbitrary distribution some discrepancies are established for analytically obtained values; improved values are obtained by large number of simulations using bisection method.

2 Basic assumptions and definitions

Let $X$ be a random variable with known cumulative distribution function (CDF) $F_X$. The characteristic value of $X$ is such value $x_\alpha$, that the probability of $X$ being less than $x_\alpha$ equals $\alpha$:

$$P[X < x_\alpha] = F_X(x_\alpha) = \alpha \quad \longrightarrow \quad x_\alpha = F_X^{-1}(\alpha). \quad (1)$$

It is obvious from (1) that the characteristic value, $x_\alpha$, depends on the distribution of the random variable. The characteristic value can be uniquely determined if the CDF is known; e.g. if its parameters are prescribed. $F_X$ is usually (directly or indirectly) described by the mean, $m_X$, and by the standard deviation, $\sigma_X$.

It is common to many practical problems that the correct values of $m_X$ and $\sigma_X$ are unknown and can only be estimated from a random sample. Thus, instead of the correct characteristic value, only its estimate could be obtained. The characteristic value estimate is itself a random variable, here denoted as $\hat{X}_\alpha$. There are a number of possibilities for the determination of the characteristic value estimate. In the present paper we are interested in the estimates for which we can control the probability $P[\hat{X}_\alpha < x_\alpha]$. Using the present approach for any previously prescribed confidence interval $\alpha_\lambda$ such characteristic value estimate, $\hat{X}_{\alpha,\lambda}$, can be determined that

$$P[\hat{X}_{\alpha,\lambda} < x_\alpha] = 1 - \alpha_\lambda. \quad (2)$$

The characteristic values and, consequently, their estimates are strongly dependent on distribution. Therefore, we need to discuss different distributions separately. Our approach is demonstrated in detail for the normal distribution and performed also for the lognormal, Gumbel, and Frechet distribution.

Usually the mean, $m_X$, and the standard deviation, $\sigma_X$, of the distribution are not known and are estimated from a random sample. Let $n$ denote the size of the sample and $X_i$ the $i$-th value of the sample. The mean and the variance of the sample are obtained from

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}, \quad S_X^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n - 1}. \quad (3)$$

$\bar{X}$ and $S_X^2$ are unbiased estimates of parameters $m_X$ and $\sigma_X$. It is obvious from (3) that $\bar{X}$ and $S_X^2$ are random variables, dependent on the distribution of $X$. The characteristic value estimate should therefore be based on the mean and the variance of the sample, because the parameters $m_X$ and $\sigma_X$ are unknown. Despite that
fact we will show that the estimates of the characteristic value can be controlled with respect to eqn. (2).

3 Normally distributed variables

The basic idea on the characteristic value determination for normally distributed variables stems from the relationship between an arbitrary normal variable $X$ and standardized normal random variable $U$

$$U = \frac{X - m_X}{\sigma_X}. \quad (4)$$

If $X$ is replaced by standardized normal random variable $U$, we obtain

$$P[X < x_\alpha] = P\left[\frac{X - m_X}{\sigma_X} < \frac{x_\alpha - m_X}{\sigma_X}\right] = F_U\left(\frac{x_\alpha - m_X}{\sigma_X}\right) = \alpha$$

and the relationship between the characteristic value and the moments of an normally distributed random variable follows:

$$x_\alpha = m_X + \sigma_X F_U^{-1}(\alpha). \quad (5)$$

Here $F_U^{-1}(\alpha)$ means the inverse of the CDF of standardized normal random variable and is thus independent on the parameters $m_X$ and $\sigma_X$.

The simple and understandable form of expression (5) represents the fundament of the estimate of the characteristic value of normally distributed random variable. If we replace the unknown parameters $m_X$ and $\sigma_X$ by their estimates, obtained from a sample, the following form for the characteristic value estimate is obtained:

$$\hat{X}_{\alpha,\lambda} = \bar{X} + \lambda S^*_X. \quad (6)$$

Note that we replaced $F_U^{-1}(\alpha)$ with parameter $\lambda$. The parameter $\lambda$ is the only free parameter in (6) and needs to be determined with respect to the previously prescribed confidence interval $\alpha_\lambda$:

$$P\left[\bar{X} + \lambda S^*_X < x_\alpha\right] = 1 - \alpha_\lambda. \quad (7)$$

If we transform the left hand side of eqn. (7) by using (5) and (6)

$$P\left[\hat{X}_{\alpha,\lambda} < x_\alpha\right] = P\left[\bar{X} + \lambda S^*_X < m_X + \sigma_X F_U^{-1}(\alpha)\right]$$

$$= P\left[\frac{\bar{X} - m_X}{\sigma_X} + \lambda \frac{S^*_X}{\sigma_X} < F_U^{-1}(\alpha)\right], \quad (8)$$

we are able to evaluate the distribution of the random variable

$$Z_\lambda = \frac{\bar{X} - m_X}{\sigma_X} + \lambda \frac{S^*_X}{\sigma_X}. \quad (9)$$

In the case of normal distribution $\bar{X}$ is normally distributed with $m_{\bar{X}} = m_X$ and $\sigma_{\bar{X}} = \sigma_X / \sqrt{n}$; the statistic $(n - 1) S^*_X^2 / \sigma_X^2$ is distributed by the chi-squared
distribution with \( n - 1 \) degrees of freedom (see e.g. [5]). After we employ the convolution integral, the probability density functions of \( Z_\lambda \) reads

\[
f_{Z_\lambda}(z) = \begin{cases} \int_{-\infty}^{0} \frac{2}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{\pi} k^{\frac{n}{2}} e^{\frac{n}{2}(z-v)^2} e^{-\frac{n}{2}(v)^2} \, dv ; & \lambda < 0 \\ \int_{0}^{\infty} \frac{2}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{\pi} k^{\frac{n}{2}} e^{\frac{n}{2}(z-v)^2} e^{-\frac{n}{2}(v)^2} \, dv ; & \lambda > 0. \end{cases}
\]

(10)

where parameter \( k \) equals \( (n-1)/2\lambda^2 \). The CDF \( F_{Z_\lambda}(z) \) is obtained by integration:

\[
F_{Z_\lambda}(z) = \int_{-\infty}^{z} f_{Z_\lambda}(\tilde{z}) \, d\tilde{z}.
\]

For further details see [6].

For the unknown \( \lambda \) and prescribed values of \( n, \alpha \) and \( \alpha_\lambda \) results are found solving one non-linear equation. In table 1 values of \( \lambda \) are shown for fixed values \( \alpha = 0.95 \) and \( \alpha_\lambda = 0.75 \) and for different values of \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>3.152</td>
<td>2.681</td>
<td>2.463</td>
<td>2.336</td>
<td>2.250</td>
<td>2.188</td>
<td>2.104</td>
<td>1.758</td>
</tr>
</tbody>
</table>

Table 1: Factor \( \lambda \) for \( \alpha = 0.95 \) and \( \alpha_\lambda = 0.75 \) and some sample sizes \( n \).

4 Lognormally distributed variables

As it has already been explained, the estimates of the characteristic values are dependent on distribution. Thus we must handle each particular distribution separately. In this section we develop the procedure for characteristic value determination for lognormally distributed variable. The basic idea of the present approach is to employ the results of the normal distribution by using its relationship to the lognormal distribution.

Lognormal random variable \( Y \) is related to normal variable \( X \) through the exponential map:

\[
Y = e^X \quad X = \ln Y.
\]

(11)

According to (11) we define the lognormal characteristic value estimate \( \hat{Y}_{\alpha,\lambda} \) as

\[
\hat{Y}_{\alpha,\lambda} = e^{\hat{X}_{\alpha,\lambda}} = e^{\bar{X} + \lambda S_X^*}.
\]

(12)

and for an arbitrary \( \alpha_\lambda \) such value of \( \lambda \) is sought to solve equation

\[
P\left[\hat{Y}_{\alpha,\lambda} < y_\alpha\right] = 1 - \alpha_\lambda.
\]

(13)

Note that the value \( \lambda \), satisfying eqn. (13), is the same as the one obtained for normal variable. For the proof see [6].

Note that in eqn. (12) the estimate of the mean and the standard deviation of the normally distributed random variable (\( \bar{X} \) and \( S_X^* \)) appear instead of the parameters of the lognormally distributed random variable (\( \bar{Y} \) and \( S_Y^* \)). The later are
directly obtained from the sample (see eqns (3). The approximative values for $\bar{X}$ and $S^*_X$ can then be obtained by several methods (see e.g. [4]). Here the method of moments is applied

$$\bar{X} = \ln \left( \frac{\bar{Y}^2}{\sqrt{(S_Y^* + \bar{Y}^2)}} \right), \quad S^*_X = \ln \left( \frac{S_Y^*}{\bar{Y}^2} + 1 \right). \quad (14)$$

By inserting (14) into (12) we obtain the final formula

$$\hat{Y}_{\alpha,\lambda} = \frac{\bar{Y}^2 e^{\lambda \sqrt{\ln \left( \frac{S^*_Y}{\bar{Y}^2} + 1 \right)}}}{\sqrt{(S_Y^* + \bar{Y}^2)}},$$

where $\bar{Y}$ and $S_Y^*$ are the mean value and the standard distribution of the sample and $\lambda$ is the same parameter as in the previous section, e.g. the parameter from table 1.

5 Variables of other known distributions

The idea for the lognormally distributed variable may be extended to an arbitrary variable with known distribution. Let $Y$ be a random variable with known distribution and $U$ a standardized normally distributed random variable. Since CDF for each variable is known, we are able to relate both random variables:

$$P[Y < y] = P[U < u]$$

$$F_Y(y) = F_U(u) \quad \rightarrow \quad y = F_Y^{-1}(F_U(u)).$$

If we denote

$$g(u) = F_Y^{-1}(F_U(u)), \quad (16)$$

it follows that $Y = g(U)$. Function $g(u)$ is also used to relate the characteristic value of $Y$ with the characteristic value of the standardized normally distributed variable:

$$\alpha = F_Y(y_\alpha) = P[Y < y_\alpha] = P[g^{-1}(Y) < g^{-1}(y_\alpha)] = P[U < g^{-1}(y_\alpha)]$$

$$= F_U(g^{-1}(y_\alpha)).$$

The use of the inverse $F_U^{-1}$ yields

$$g^{-1}(y_\alpha) = F_U^{-1}(\alpha) \quad \rightarrow \quad y_\alpha = g\left(F_U^{-1}(\alpha)\right). \quad (17)$$

The same relationship is applied for the definition of characteristic value estimate $\hat{Y}_{\alpha,\lambda}$:

$$\hat{Y}_{\alpha,\lambda} = g\left(\frac{\bar{X} - m_X}{\sigma_X} + \lambda \frac{S^*_X}{\sigma_X}\right) = g\left(Z_\lambda\right). \quad (18)$$

Since $Z_\lambda$ is the same as in (9) the parameter $\lambda$ equals the results for the normally distributed variable.
Further approximation is made because the values \( m_X \) and \( \sigma_X \) are unknown. If we assume \( m_X \approx \bar{X} \) and \( \sigma_X \approx S_X^* \), the estimate (18) simplifies considerably:

\[
\hat{Y}_{\alpha,\lambda} = g(\lambda).
\]

### 5.1 Gumbel distribution

As a special case we analyze the Gumbel distribution. The CDF for Gumbel distribution is

\[
F_Y(y) = \exp \left( -\exp [-\beta(y - w)] \right).
\]

Here, \( \beta \) and \( w \) are the parameters of the distribution, dependent solely on \( m_Y \) and \( \sigma_Y \):

\[
\beta = \frac{\pi}{\sigma_Y \sqrt{6}}, \quad w = m_Y - \frac{\gamma}{\beta}, \quad \gamma \approx 0.577216.
\]

The CDF of the standardized normal distribution is expressed with the error function as

\[
F_U(u) = \frac{1}{2} \left( 1 + \text{Erf} \left( \frac{u}{\sqrt{2}} \right) \right).
\]

By inserting (20) and (21) into (16) we obtain

\[
g(u) = m_Y - \frac{\sigma_Y \sqrt{6}}{\pi} \left[ \gamma + \ln \left( -\ln \left[ \frac{1}{2} \left( 1 + \text{Erf} \left( \frac{u}{\sqrt{2}} \right) \right) \right] \right) \right].
\]

After approximating \( m_Y \) and \( \sigma_Y \) with \( \bar{Y} \) and \( S_Y^* \), we can estimate the characteristic value of the Gumbel distribution as

\[
\hat{Y}_{\alpha,\lambda} = \bar{Y} - \frac{S_Y^* \sqrt{6}}{\pi} \left[ \gamma + \ln \left( -\ln \left[ \frac{1}{2} \left( 1 + \text{Erf} \left( \frac{\lambda}{\sqrt{2}} \right) \right) \right] \right) \right].
\]

### 5.2 Frechet distribution

The CDF for Frechet distribution is

\[
F_Y(y) = \exp \left( -\left( \frac{w}{y} \right)^k \right),
\]

\( k \) and \( w \) are the parameters of the distribution, dependent solely on \( m_Y \) and \( \sigma_Y \) (for details see [1]). The relationship of Frechet distribution to standardized normal distribution is obtained by inserting (24) and (21) into (16):

\[
g(u) = \frac{m_Y}{\Gamma \left( 1 - \frac{1}{k} \right)} \left( -\ln \left[ \frac{1}{2} \left( 1 + \text{Erf} \left( \frac{u}{\sqrt{2}} \right) \right) \right] \right)^{-\frac{1}{k}}.
\]

After approximating \( m_Y \) with \( \bar{Y} \) and \( k \) with \( \bar{k} \) by employing \( S_Y^* \) of the sample, we can estimate the characteristic value of the Frechet distribution as

\[
\hat{Y}_{\alpha,\lambda} = \frac{\bar{Y}}{\Gamma \left( 1 - \frac{1}{\bar{k}} \right)} \left( -\ln \left[ \frac{1}{2} \left( 1 + \text{Erf} \left( \frac{\lambda}{\sqrt{2}} \right) \right) \right] \right)^{-\frac{1}{\bar{k}}}. 
\]
Table 2: Estimated probability $P \left[ \hat{X}_{\alpha,\lambda} < x_\alpha \right]$ for normal distribution.

<table>
<thead>
<tr>
<th>n</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P[\hat{X}<em>{\alpha,\lambda} &lt; x</em>\alpha]$</td>
<td>0.2498</td>
<td>0.2502</td>
<td>0.2501</td>
<td>0.2498</td>
<td>0.2503</td>
<td>0.2499</td>
</tr>
</tbody>
</table>

6 Simulations of the characteristic value determination

A huge number of repetitions of the sample selections can easily be simulated by computer using a random number generator. In computer simulations we can prescribe the values of mean and standard deviations in contrast to practical sampling where these parameters are usually unknown.

The simulations were performed by the following algorithm:

Reading the input values of $m_X$ and $\sigma_X$.
Calculation of the exact characteristic value $x_\alpha$.
Loop over simulations.
  Loop over elements of the sample.
    Random number generation.
  End loop.
Mean and standard deviance estimate (eqns (3)).
Calculus of the estimate $\hat{X}_{\alpha,\lambda}$ using eqn. (6) and $\lambda$ values from table 1.
End loop.
Estimation of probability $P \left[ \hat{X}_{\alpha,\lambda} < x_\alpha \right]$.

The estimation of probability $P \left[ \hat{X}_{\alpha,\lambda} < x_\alpha \right]$ is obtained by counting the number of estimates $\hat{X}_{\alpha,\lambda}$ which are less than $x_\alpha$ and by dividing this number by the number of simulations. In the same way the simulations for other distributions were done, only random number generator and estimate characteristic value formula were taken with respect to different distribution. For the lognormal distribution the estimate $\hat{Y}_{\alpha,\lambda}$ was obtained by eqn. (15); for others eqn. (19) was used.

Results for normal distribution with $m_X = 30$, $\sigma_X = 6$ obtained by 1 000 000 simulations are shown in table 2. From table 2 follows that our parameters $\lambda$ proved to be very accurate. Estimates differ from the expected values only at the fourth significant digit. For greater number of simulations even more accurate results are expected.

Results of the 1 000 000 simulations for the lognormal population with $m_Y = 30$ and $\sigma_Y = 6$ are shown in table 3. The results in table 3 are not as accurate as those in table 2 because of the assumption (14). However, the method of the estimation of the characteristic values of lognormal random variable, described in this paper, is sufficiently precise for practical use.
Table 3: Estimated probability $P[\hat{Y}_{\alpha,\lambda} < y_\alpha]$ for lognormal distribution.

<table>
<thead>
<tr>
<th>$n$</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P[\hat{Y}<em>{\alpha,\lambda} &lt; y</em>\alpha]$</td>
<td>0.2604</td>
<td>0.2615</td>
<td>0.2636</td>
<td>0.2637</td>
<td>0.2642</td>
<td>0.2653</td>
</tr>
</tbody>
</table>

The same number of simulations and the same values of mean and standard deviation were used for Gumbel distribution. Results are shown in table 4. We can observe that the results are probability $P[\hat{Y}_{\alpha,\lambda} < y_\alpha]$ increases with increasing sample size.

Table 4: Estimated probability $P[\hat{Y}_{\alpha,\lambda} < y_\alpha]$ for Gumbel distribution and $\lambda$ from table 1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P[\hat{Y}<em>{\alpha,\lambda} &lt; y</em>\alpha]$</td>
<td>0.237</td>
<td>0.245</td>
<td>0.251</td>
<td>0.255</td>
<td>0.260</td>
<td>0.270</td>
</tr>
</tbody>
</table>

Finally, the same results are shown for Frechet distribution. Results are shown in table 5. We can see that the deviation from the required value 0.25 increase with increasing coefficient of variation $V_Y$.

Table 5: Estimated probability $P[\hat{Y}_{\alpha,\lambda} < y_\alpha]$ for Frechet distribution and $\lambda$ from table 1.

<table>
<thead>
<tr>
<th>Sample size $n$</th>
<th>$V_Y$ 5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>0.246</td>
<td>0.256</td>
<td>0.263</td>
<td>0.267</td>
<td>0.273</td>
<td>0.284</td>
</tr>
<tr>
<td>0.10</td>
<td>0.255</td>
<td>0.267</td>
<td>0.274</td>
<td>0.279</td>
<td>0.286</td>
<td>0.300</td>
</tr>
<tr>
<td>0.25</td>
<td>0.283</td>
<td>0.298</td>
<td>0.309</td>
<td>0.316</td>
<td>0.325</td>
<td>0.352</td>
</tr>
<tr>
<td>0.50</td>
<td>0.324</td>
<td>0.345</td>
<td>0.359</td>
<td>0.369</td>
<td>0.383</td>
<td>0.439</td>
</tr>
</tbody>
</table>

7 Characteristic value parameters obtained by simulations

The error of the analytically obtained values for Gumbel distribution, is the motivation to improve the method. Without the assumption $m_X \approx \bar{X}$ and $\sigma_X \approx S_X^*$ analytical evaluation is to demanding, thus simulations were used to obtain more precise values for $\lambda$. The algorithm similar to the one in previous section has been used. In order to improve $\lambda$ we used the bisection until the estimated probability $P[\hat{Y}_{\alpha,\lambda} < y_\alpha]$ for 1 000 000 simulations was equal to 0.25 in 4 significant digits.
As a consequence of improvement, new table of correction factors is obtained to be used only for the Gumbel distribution. The corrected parameters $\lambda$ for Gumbel distribution are shown in table 6.

Table 6: Correction factor $\lambda$ for Gumbel distribution for $\alpha = 0.95$ and $\alpha_{\lambda} = 0.75$ as obtained by simulations.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2.425</td>
</tr>
<tr>
<td>6</td>
<td>2.325</td>
</tr>
<tr>
<td>7</td>
<td>2.254</td>
</tr>
<tr>
<td>8</td>
<td>2.199</td>
</tr>
<tr>
<td>10</td>
<td>2.122</td>
</tr>
<tr>
<td>100</td>
<td>1.766</td>
</tr>
</tbody>
</table>

The simulation based correction factors for the Frechet distribution are even more demanding for evaluation. The reasons are the nonlinear dependence between $k$ and $\sigma_Y$ and, consequently, the nonlinear relationship between $\hat{Y}_{\alpha,\lambda}$ and $k$. The nonlinearity of $\hat{Y}_{\alpha,\lambda}$ with respect to $k$ results in the parameter $\lambda$ to be dependant on the coefficient of variation as well as on the sample size. By simulations and iteratively by using bisection parameters $\lambda$ were evaluated for several sample sizes and several coefficients of variation. Similar to Gumbel distribution in eqn. (26) the values from table 7 instead of the ones from table 1 should be used in (26) to obtain the 75% confidence interval.

Table 7: Correction factor $\lambda$ for Frechet distribution for $\alpha = 0.95$ and $\alpha_{\lambda} = 0.75$ and various values of the coefficient of variation as obtained by simulations.

<table>
<thead>
<tr>
<th>$V_Y$</th>
<th>Sample size $n$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5</td>
</tr>
<tr>
<td>0.05</td>
<td>2.451</td>
</tr>
<tr>
<td>0.05</td>
<td>2.477</td>
</tr>
<tr>
<td>0.05</td>
<td>2.551</td>
</tr>
<tr>
<td>0.05</td>
<td>2.648</td>
</tr>
</tbody>
</table>

8 Conclusions

Determination of the characteristic values from small samples was analyzed for several different distributions. The main points of the present approach are as follows:

(i) For normal distribution exact analytical formulation of the problem can be found. Analytical derivation results in a one-dimensional non-linear formula for determination of parameters.

(ii) These parameters are used directly with the estimates of mean and standard deviation from the sample to evaluate the estimate of the characteristic value with previously prescribed confidence interval. Analytical results are confirmed by simulations.
(iii) Lognormal distribution is directly connected to normal distribution through the exponential map. This relationship allows us to extended the formal algorithm from the normal to lognormal distribution.

(iv) For other distributions relation to normal distribution is more complicated. In order to obtain analytical expressions additional assumptions must be taken. They result in less accurate results. For better results new tables of parameters are prepared numerically by the use of simulations.

(v) In the case of Gumbel distribution the correction factors $\lambda$ depend only on sample size $n$. On the other hand, the correction factors for Frechet distribution depend on sample size $n$ and coefficient of variation $V_Y$.

References