Crack modelling: an aid to optimisation through the analysis of stress concentrators
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Abstract

Engineering components and structures invariably fail from stress concentrations: geometrical features which give rise to high local stresses, known as "hot spots". This paper is concerned with a new method for predicting the behaviour of these hot spots, using data from finite element (FE) analysis and other numerical techniques. The method is of interest in optimisation because it provides a measure of the severity of a hot spot with respect to fatigue failure.

The problem is non-trivial because the stress at the hot spot and its fatigue behaviour are not simply correlated. Reasons for this include: (a) plasticity which alters the stress and strain at the hot spot; (b) stressed volume: small notches are not as dangerous as large ones; (c) stress gradients: high stress gradient improves fatigue life; (d) material differences: some materials are less sensitive to notches than others; and (e) saturation: fatigue behaviour becomes insensitive to stress at high stress concentrations. One effect of this is that failure may occur from a point on the component which is not the point where the stress is highest, and design optimisations which reduce the hot-spot stress may not improve fatigue life.

Current methods use elastic-plastic analysis to overcome (a), and empirical factors to allow for (b) and (c). These methods are difficult to interface with FE analysis and have a high degree of uncertainty. The new method makes use of the theory of linear elastic fracture mechanics (LEFM), which is designed to describe the behaviour of cracks using elastic analysis of the crack-tip singularity. This has been adapted to analyse stress concentrations, producing a parameter which is equivalent to the stress intensity parameter, K, of LEFM.
The method has been tested on a component for which design changes were made in order to avoid failure in service. Two different design changes were implemented; the change which was most effective in reducing the hot-spot stress was not the most effective in preventing failure in service. This behaviour was correctly predicted by the new method of analysis, with an accuracy of about 10% on stress.

**Introduction**

Prevention of failure is a major factor in the optimisation of a structure. There are many modes of failure but fatigue is the largest single cause of catastrophic failure in engineering components. Thus 'fatigue design' must figure highly in any programme of optimisation. Though there are some notable exceptions (e.g. aircraft), most components which experience cyclic stress are required to endure very large numbers of cycles - in excess of one million - and consequently their behaviour falls into the category of 'high-cycle fatigue', which is distinguished by nominally-elastic stresses and a strong relationship between applied stress and number of cycles to failure. In fact, most components used in, for example, the automotive, machine-tool and medical-device industries are designed to operate below their fatigue limit. Improvements often take the form of weight-reduction or the adoption of a new material. In such circumstances, accurate prediction of the fatigue limit of the component is crucial.

Failures invariably occur from stress concentrations (hot-spots), and the behaviour of material at the hot spot - in particular the tendency for a fatigue crack to initiate and grow - differs from behaviour in the bulk as outlined above. This behaviour has been recognised for many years: for example, Neuber [1] described a method for allowing for the effect of notch size via a consideration of the 'stressed volume'. Empirical laws, some of them half a century old, predominate even in modern software. Methods based on the consideration of plasticity at the hot spot reduce, but do not entirely eliminate this empiricism, and require the complexity of elastic-plastic analysis [see, for example, Suresh [2]. The aim of the present paper is to describe a new method which has been designed to analyse the fatigue behaviour of stress concentrators in components, using data from elastic FE analysis. A case history is presented, and the potential for using the method for optimisation is discussed.

**Basic Theory: The Problem Solved for Notched Specimens**

Fig.1 encapsulates the problem which we are facing. Imagine a series of notched specimens in which the notch depth, D, is kept constant but the root radius, ρ, is progressively reduced, thus increasing the severity of the hot spot. This can be quantified by the stress concentration factor, $K_t$: the ratio of
hot-spot stress to nominal applied stress. At $K_t=1$ we have a plain (unnotched) specimen whose fatigue limit, $\Delta \sigma_{on}$, is a material property. As $K_t$ is increased one might expect the fatigue limit of the notched specimen, $\Delta \sigma_{on}$, to reduce in proportion to $K_t$, and initially this is indeed what happens. However, as $K_t$ increases, behaviour deviates from this expectation, so that at large values of $K_t$ the prediction so obtained would be highly conservative. The shape of the experimental curve depends also on notch depth and on material properties; many empirical laws have been developed to try to bridge the gap between the $K_t$-based prediction and experimental behaviour. However, the problem was solved twenty years ago by Smith and Miller [3] who noted that the diagram can be split into two regions, above and below a critical value, $K_t^*$. Below this value, $\Delta \sigma_{on}$ is well predicted by $\Delta \sigma_{o}/K_t$, and above $K_t^*$ it is constant, independent of $K_t$. In this region $\Delta \sigma_{on}$ can be estimated by calculating the behaviour of a crack of the same depth. The crack is simply the extreme case of our notch when $\rho=0$. This cannot be analysed on the basis of $K_t$ (which would be infinite), but can be treated using linear-elastic fracture mechanics (LEFM) which defines a stress intensity, $K$, through equations of the form:

$$K = F\sigma(\pi D)^{1/2}$$  \hspace{1cm} (1)$$

...where $D$ is the length of the crack and $F$ is a constant which depends on the geometry of crack and specimen. A fatigue crack will not propagate if the
cyclic stress intensity, $\Delta K$, is less than a threshold value, $\Delta K_{th}$, which is a material property. Thus eqn. 1 can be used to predict $\Delta \sigma_{on}$ by:

$$\Delta \sigma_{on} = \Delta K_{th}/[F(\pi D)^{1/2}]$$

(2)

This leads to a horizontal line on fig. 1, and it is remarkable how closely the experimental data conform to this simple prediction. In practice there is a degree of conservatism which varies from zero (at the two ends of the graph) to about 20% - this in itself is a useful feature for the designer. The relative position of the two prediction lines varies considerably for different materials, since it depends on the relative values of $\Delta \sigma_0$ and $\Delta K_{th}$. It is notable that, for low-strength materials such as cast iron and mild steel, $K_t*$ may be less than 3, which means that many design features in components will fall into the upper region, i.e. they can be described as 'crack-like' for the purposes of fatigue analysis.

**Extension of the Theory for use in Components**

This method works very well for specimens containing notches of standard geometry, but it is difficult to extend it to arbitrary shapes and thus to real components. The reason is that there is no equivalent of eqns 1 and 2: the constant $F$ will be unknown and, more fundamentally, there will be no equivalent of the length parameter, $D$ in a shape such as, for example, a corner or bend. This problem is solved when we realise that the basic theory is essentially a modelling exercise: the actual notch geometry is modelled as a crack. When dealing with a stress-concentrator of complex geometry (e.g. a fillet in an automotive crankshaft) the problem is reduced to one of finding the crack to which our stress-concentrator is equivalent. This can be done by interrogating the stress field produced by an FE analysis of the component. Fig. 2 summarises the method, which has been termed "crack modelling" [4,5].

The stress field of the component (shown top-left in the figure) is compared with that of a model crack (top-right); the crack geometry chosen is a simple straight crack loaded in tension. Consider the plot of stress versus distance, $r$, measured along a line moving outwards from the hot-spot. This can be compared with a similar plot for the crack. Two parameters (applied stress and crack length) can be varied in the crack model until a best fit is achieved between the two stress/distance curves. We have now found the crack to which this component is equivalent.

Since the method uses only an elastic FE analysis it is simple to find the loadings on the component for which $\Delta K$ (calculated from the model crack using eqn.1) is equal to $\Delta K_{th}$. These loadings correspond to the fatigue limit of the component. There is one qualification here, which is that the method only applies if $K_t> K_t*$ (fig. 1). This can be checked by making a second calculation,
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Component FEA
Applied Loads, \( L \)

Centre-Cracked
Infinite Plate

Stress
Applied Loads \( L \)

Stresses along \( X-X' \)

Stress
Applied Stress Intensity, \( K \)

Stresses along \( Y-Y' \)

Best fit gives a \( K \) prediction corresponding to loads \( L \)

Figure 2: The crack-modelling approach
equivalent to the use of $K_t$ for notches, in which the hot-spot stress range is compared with $\Delta \sigma_o$.

**The Failure Point May Not be the Worst Hot-Spot**

One interesting consequence of this theory is that, given a component containing several hot-spots, the one which shows the highest local stress value may not be the one from which failure occurs. For example, consider a bar in tension containing two surface notches: notch A is a semi-circle, $D=\rho=10\text{mm}$; notch B is a sharper, semi-elliptical shape with $D=5\text{mm}$, $\rho=1\text{mm}$. Applied loading which causes a stress of 100MPa at A will cause about 180MPa at B. However, if both of the these notches are 'crack-like', which they will be, for example, in a typical cast iron, then failure will occur preferentially from A, because it has a higher $D$ value and therefore a higher stress intensity (eqn.1), even though B shows a much higher local stress. This effect is of course crucial in optimisation: efforts to improve this component by reducing the worst hot spot will have no effect on its fatigue behaviour. Methods which allow for local plasticity, such as the commonly-used Neuber approach, will not detect this problem.

**A Case Study**

The method has been applied successfully to a number of components (e.g. [4,5]). Here we describe its use in considering design changes in a component which was experiencing fatigue failure in service [4]. For commercial reasons the nature of this component is confidential; suffice it to say that it was a large casting used for marine applications. A number of these components failed in use due to fatigue cracking from a sharp corner with a fillet radius of 0.3mm. This original design will be referred to as S1. Subsequently two design changes were attempted. In design S2 the fillet radius was increased to 3.2mm, but it was found that failure still occurred. In design C the original fillet was retained, but service loads were reduced by about 14%; in this case no failures occurred.

Fig.3 shows stress-distance curves for the three designs obtained from FE modelling; the maximum principal stress is given as a function of distance from the fillet when the maximum cyclic load is being applied. Condition S2 reduces the hot spot from 1279MPa to 384MPa, but the seemingly more modest reduction of design C was in fact more successful. Table I summarises the results of a crack-modelling analysis, comparing the estimated values of $\Delta K$ with the threshold, $\Delta K_{\text{th}}$, for each design. An extra feature of this problem is that, in design C, there is also a change to the load ratio, $R$, defined as the ratio of minimum to maximum stress. Since $\Delta K_{\text{th}}$ varies with $R$ we use two slightly different values, obtained experimentally. Values of $\Delta K_{\text{th}}$ as a function of $R$ are available for a wide range of materials.
Figure 3: Stress-distance curves for three component designs

Table I: Results of Crack Modelling for the Three Component Conditions

<table>
<thead>
<tr>
<th>Condition</th>
<th>Estimated K Values, MPa(m)^{1/2}</th>
<th>R</th>
<th>Measured Threshold</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K_{\text{max}}$</td>
<td>$K_{\text{min}}$</td>
<td>$\Delta K$</td>
</tr>
<tr>
<td>S1</td>
<td>16.50</td>
<td>9.22</td>
<td>7.28</td>
</tr>
<tr>
<td>S2</td>
<td>14.13</td>
<td>7.83</td>
<td>6.30</td>
</tr>
<tr>
<td>C</td>
<td>14.25</td>
<td>9.22</td>
<td>5.03</td>
</tr>
</tbody>
</table>

The analysis predicts failure in the original design (S1), since the value of $\Delta K$ is significantly greater than $\Delta K_{\text{th}}$. We also correctly predict that design C will be safe. In the case of S2, the $\Delta K$ value is less than the threshold, but only by a margin of 7.8%, so we would conclude that design S2 is in some danger of failing. What is perhaps more important as regards optimisation is that the method was able to place the three designs in order of safety, with C being the safest and S1 the most dangerous. The increase of fillet radius in S2, which in many cases would be in accordance with 'best practice' in fatigue design, is found to be ineffective here because we are in the 'crack like' regime where $\rho$ is unimportant.
Stress-Field Accuracy and Mesh Density

The method is, naturally, dependant on the accuracy of the FE analysis from which it proceeds. However, it is not strongly dependant on an accurate prediction of the hot-spot stress, and in this respect it differs from other methods of fatigue analysis which have been incorporated into post-processing software. As an illustration of this, we consider the range of values of r over which the curves are optimised (see fig.2). Imagine a portion of the stress/distance curve, from \( r_{\text{min}} \) to \( r_{\text{max}} \). The value of \( r_{\text{min}} \) cannot be zero since at this point the stress in the crack model tends to infinity; likewise \( r_{\text{max}} \) is limited by the physical size of the component and by other considerations: for example, if bending loads are applied then the stress will become negative in parts of the component, but not in the crack model. However, it is found that the prediction of K, and therefore the accuracy of the crack-modelling method,

![Graph](image)

Figure 4: Effect of changing \( r_{\text{min}} \) and \( r_{\text{max}} \) on K estimates for a sharp notch loaded in tension. K is measured in MPa(m)^{1/2} and r in mm.
is not strongly dependant on these limits of \( r \). For example, fig.4 shows the effect of changing \( r_{\text{min}} \) and \( r_{\text{max}} \) for a V-shaped notch of depth 2.14mm and root radius 0.1mm. In the upper diagram, \( r_{\text{min}} \) is varied, keeping \( r_{\text{max}} = 8 \text{mm} \); in the lower diagram \( r_{\text{min}} \) is held at a value close to zero. There is remarkably little change in the estimate of \( K \), even when \( r_{\text{min}} \) is increased beyond the dimensions of the notch. This is due to the fact that the estimate of \( K \) is reached by considering the whole stress field, and so it is not sensitive to changes at the hot-spot. A consequence of this is that mesh-refinement in the region of the notch, whilst it will greatly improve the description of near-notch stresses and thus is essential for other methods of fatigue analysis, is not important in this case. Analysis of an automotive crankshaft (previously reported in [5]) has shown that mesh refinement, which increased hot-spot stresses by 30\%, had no significant effect on the \( K \)-estimate.

Concluding Remarks

The crack-modelling technique, though still in its infancy, has great potential for use in the optimisation of design in engineering components. It interfaces readily with FE software, can be used with 2D and 3D models, and requires only an elastic stress analysis. Because it is based on a full-field analysis borrowed from LEFM theory, it is not sensitive to the description of stress in the region close to the stress-concentration, and therefore does not require a large amount of mesh-refinement. It requires relatively few material parameters, including the stress intensity threshold for crack propagation, \( \Delta K_{\text{th}} \), which is now commonly available. It can be used with a wide range of materials, though its range of validity is much larger in low-strength materials such as mild steels and cast irons.

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References


