



Bounds and homogenization of some optimal reiterated honeycombs

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Abstract

We consider reiterated honeycomb-structures with m different micro-levels. By means of the homogenization theory we obtain upper and lower bounds for the corresponding effective properties. These bounds turn out to be very close to each other for large values of the reiteration number m . In fact, our results show that they converge to the same limit as m goes to infinity. Moreover, we point out that this limit is optimal within the class of two-phase structures with prescribed volume fractions. We also present some numerical results for the case $m = 1$.

1. Introduction

Many problems in the homogenization theory deal with the characterization of sets of admissible effective properties generated from classes of structures and composites with given local properties and volume fractions (see e.g. Avellaneda [1], Francfort & Murat [5], James et al. [7] and Milton [15]). Such sets have been successfully found in many cases. For example for the two-dimensional two-phase conductivity problem the set of pairs (λ_1, λ_2) (λ_i being the effective conductivity in the direction x_i direction) can be illustrated as in Figure 1.1. Structures corresponding to the upper or lower boundary of such sets are naturally referred to as optimal. There exist several examples of optimal structures, e.g. the *Hashin-Shtrikman coated sphere-assemblages* and the *multi-rank laminate-structure* (see Milton [14] and Tartar [17]).

A fairly new structure called the *reiterated cell-structure* (introduced in Lukkassen [10]) has also proven to be optimal both for linear conductivity

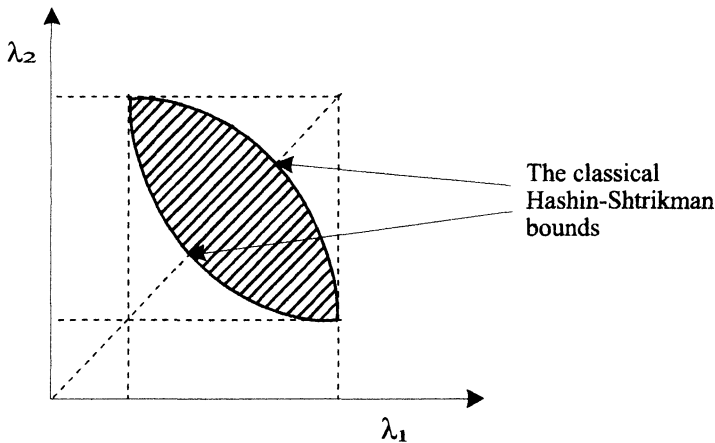


Figure 1.1: The set of admissible effective parameters.

and elasticity problems. Moreover, one can find nonlinear examples where this structure possesses effective properties which are more extreme than that of any multi-rank laminate-structure (see Braides & Lukkassen [3]).

In this paper we consider a similar structure of m different micro-levels obtained by reiterating regular honeycombs. Motivated by the ideas in [10] we obtain upper and lower bounds for the corresponding effective properties. These bounds turn out to be very close to each other for large values of the reiteration number m . In fact, our results show that they converge to the same limit as m goes to infinity. Moreover, we point out that this limit is optimal within the class of two-phase structures with prescribed volume fractions.

For the reiterated cell-structure the corresponding bounds were obtained by reiterating the estimates presented in Lukkassen [9]. We remark that, for reiterated honeycombs these estimates are useless, and this requires a deep change in our proof.

The paper is organized as follows. Our main result are presented and discussed in §2. In §3 we give the proof of our main result. In §4 we present some numerical experiments for the case $m = 1$. Some final comments can be found in §5.

2. The main result

In this section we present the main result of this paper. We first introduce the *reiterated honeycomb-structure*, described by characteristic func-

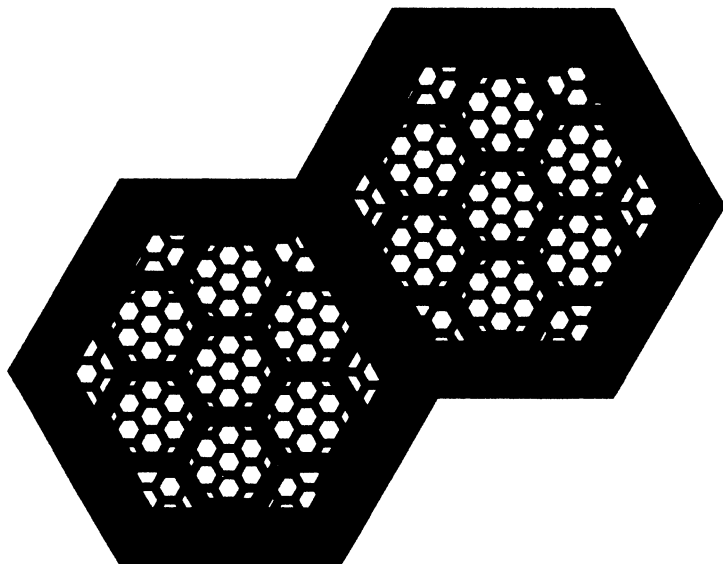


Figure 1.2: The reiterated honeycomb-structure for $m = 3$.

tions $\chi_{hon,h}$ of the form

$$\chi_{hon,h}(x) = \chi(hx, h^2x, \dots, h^m x).$$

Let $0 \leq v \leq 1$. With $\mathfrak{S}_v$ we will denote the family of all sequences $\{\chi_h\}$ of characteristic functions on $\mathbf{R}^2$ such that the mean value $\langle \chi_h \rangle$ of χ_h converges to v as $h \rightarrow \infty$. Let $m \in \{1, 2, 3, \dots\}$ and let A_m denote the interior a regular honeycomb with relative wall thickness b_m (see Figure 3.1) given by

$$b_m = 1 - v^{1/2m}.$$

Moreover, let χ_m denote the characteristic function of the set A_m and put

$$\chi_{hon,h}(x) = \prod_{i=1}^m \chi_m(h^i x), \quad h \in \{1, 2, 3, \dots\}.$$

The sequence $\{\chi_{hon,h}\}_{h=1}^\infty$ will hereafter be called the reiterated honeycomb-structure. It is easily seen that the mean value $\langle \chi_{hon,h} \rangle$ of $\chi_{hon,h}$ converges to v . Consequently, $\{\chi_{hon,h}\} \in \mathfrak{S}_v$.

The main result of this paper reads as follows:

Theorem 1. Assume that the local conductivity a_m is given by

$$a_m(x) = k_1(1 - \chi_{hon,h}(x)) + k_2\chi_{hon,h}(x)$$

for some constants k_1 and k_2 . Then, the corresponding effective conductivity converges to a number q_m as $h \rightarrow \infty$ satisfying

$$\frac{1}{q_m - k_1} = \frac{1}{k_2 - k_1} \frac{1}{v} + \frac{r_m}{1 + v^{\frac{1}{2m}}} \frac{1 - v}{k_1 v},$$

where $v^{\frac{1}{2m}} \leq r_m \leq 1$.

Remark 1. As a special result of the above theorem we obtain that

$$q_m \rightarrow k_1 + \left(\frac{1}{k_2 - k_1} \frac{1}{v} + \frac{1 - v}{2k_1 v} \right)^{-1},$$

as $m \rightarrow \infty$ which is precisely the Hashin-Shtrikman upper bound for $k_2 < k_1$ and the Hashin-Shtrikman lower bound if $k_2 > k_1$ (compare e.g. with [8]). We recall that Hashin-Shtrikman bounds are best possible within the class of macroscopically isotropic structures contained in $\mathfrak{F}_v$.

3. Proof of the main result

Let $a(x)$ be a local conductivity given by $a(x) = k_2 \chi_m(x) + k_1(1 - \chi_m(x))$. According to the homogenization method (see e.g. the books [2, 4, 16]) the corresponding effective conductivity q satisfies the variational formula

$$q = \min_{v \in W_{\text{per}}^{1,2}(Y)} \left\{ \frac{1}{|Y|} \int_Y a(x) \left(\frac{\partial v}{\partial x_1} + 1 \right)^2 + a(x) \left(\frac{\partial v}{\partial x_2} \right)^2 dx \right\}, \quad (3.1)$$

where Y denotes the rectangle which gives the periodicity of honeycomb-structure (see Figure 3.1) and $W_{\text{per}}^{1,2}(Y)$ is the space of Y -periodic functions in the usual Sobolev space $W^{1,2}(Y)$. Moreover, by Jikov [8, p. 39] we have the following equivalent variational formula:

$$\frac{1}{q} = \min_{v \in W_{\text{per}}^{1,2}(Y)} \left\{ \frac{1}{|Y|} \int_Y \frac{1}{a(x)} \left(\frac{\partial v}{\partial x_1} + 1 \right)^2 + \frac{1}{a(x)} \left(\frac{\partial v}{\partial x_2} \right)^2 dx \right\}. \quad (3.2)$$

Now, let the honeycomb-structure be divided into the regions 1,...,6 as illustrated in Figure 3.2 and let $w \in W_{\text{per}}^{1,2}(Y)$ be defined such that

$$(\text{grad } w)^T = \begin{cases} (1, 0) & \text{in 1 and 6} \\ \left(\frac{-3+4b_m}{4b_m}, -\frac{\sqrt{3}}{4b_m} \right) & \text{in 2} \\ \left(\frac{-3+4b_m}{4b_m}, \frac{\sqrt{3}}{4b_m} \right) & \text{in 3} \\ \left(\frac{-1+b_m}{b_m}, 0 \right) & \text{in 4 and 5} \end{cases}.$$

By (3.1) and (3.2) we obtain that

$$q^- \leq q \leq q^+, \quad (3.3)$$

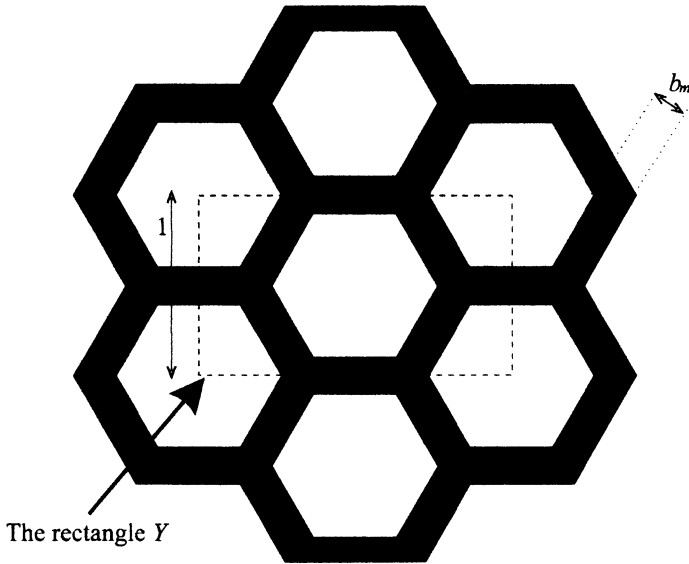


Figure 3.1: The classical honeycomb-structure.

where

$$q^+ = \min_{t \in \mathbf{R}} \left\{ \frac{1}{|Y|} \int_Y a(x) \left(\frac{\partial(tw)}{\partial x_1} + 1 \right)^2 + a(x) \left(\frac{\partial(tw)}{\partial x_2} \right)^2 dx \right\}$$

and

$$\frac{1}{q^-} = \min_{t \in \mathbf{R}} \left\{ \frac{1}{|Y|} \int_Y \frac{1}{a(x)} \left(\frac{\partial(tw)}{\partial x_1} + 1 \right)^2 + \frac{1}{a(x)} \left(\frac{\partial(tw)}{\partial x_2} \right)^2 dx \right\}.$$

These values are easily found explicitly. In fact, we obtain that

$$q^+ = k_1 \frac{k_1 b_m + k_2 - k_2 b_m}{-k_1 b_m + k_1 b_m^2 + k_1 + k_2 b_m - k_2 b_m^2}$$

and

$$q^- = \left(k_1^{-1} \frac{k_1^{-1} b_m + k_2^{-1} - k_2^{-1} b_m}{-k_1^{-1} b_m + k_1^{-1} b_m^2 + k_1^{-1} + k_2^{-1} b_m - k_2^{-1} b_m^2} \right)^{-1}.$$

Accordingly, since $b_m = 1 - v^{1/2m}$, we obtain that

$$(q^+ - k_1)^{-1} = v^{-\frac{1}{m}} (k_2 - k_1)^{-1} + v^{1/2m} \frac{1 - v^{1/2m}}{k_1 v^{1/m}}, \quad (3.4)$$

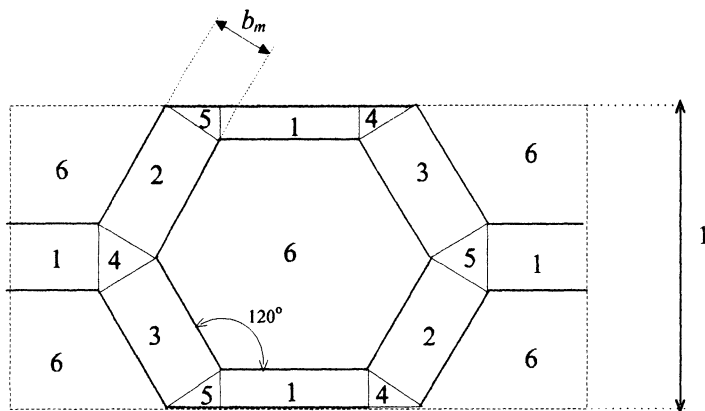


Figure 3.2: Regions 1,...,6.

$$(q^- - k_1)^{-1} = v^{-\frac{1}{m}} (k_2 - k_1)^{-1} + \frac{1 - v^{1/2m}}{k_1 v^{1/m}}. \quad (3.5)$$

Moreover, by to the iterated homogenization theorem (see [2]) it holds that $q_m = q_{m,m}$, where $q_{m,m}$ is determined by the following iterative variational formula:

$$\begin{aligned} q_{m,i} &= \min_{v \in W_{\text{per}}^{1,2}(Y)} \left\{ \frac{1}{|Y|} \int_Y b_i(x) \left(\frac{\partial v}{\partial x_1} + 1 \right)^2 + b_i(x) \left(\frac{\partial v}{\partial x_2} \right)^2 dx \right\}, \\ b_i(x) &= q_{m,i-1} \chi_m(x) + k_1 (1 - \chi_m(x)), \quad i = 1, \dots, m, \\ q_{m,0} &= k_2. \end{aligned}$$

Thus, using (3.3), (3.4) and (3.5) we find that

$$q_m^- \leq q_m \leq q_m^+, \quad (3.6)$$

where q_m^+ and q_m^- are determined by the following iterative formulae:

$$(q_i^+ - k_1)^{-1} = v^{-\frac{1}{m}} (q_{i-1}^+ - k_1)^{-1} + v^{1/2m} \frac{1 - v^{1/2m}}{k_1 v^{1/m}}, \quad (3.7)$$

$$(q_i^- - k_1)^{-1} = v^{-\frac{1}{m}} (q_{i-1}^- - k_1)^{-1} + \frac{1 - v^{1/2m}}{k_1 v^{1/m}}, \quad (3.8)$$

$$q_0^+ = q_0^- = k_2.$$

Moreover, if $a, b \in \mathbf{R}$ and $\{x_i\}_{i=1}^m$ is a sequence satisfying the recursive formula

$$x_i = ax_{i-1} + b,$$

then

$$x_i = a^i x_0 + \frac{1 - a^i}{1 - a} b.$$

Hence, it follows that

$$(q_m^+ - k_1)^{-1} = \frac{1}{v} (k_2 - k_1)^{-1} + \frac{v^{1/2m}}{(1 + v^{1/2m})} \frac{1 - v}{k_1 v},$$

$$(q_m^- - k_1)^{-1} = \frac{1}{v} (k_2 - k_1)^{-1} + \frac{1}{(1 + v^{1/2m})} \frac{1 - v}{k_1 v},$$

and, by (3.6) the theorem follows.

4. Numerical computations

In this section we present some numerical computations of the effective conductivity q_m for the case $m = 1$. We also compare these values with the theoretical (and optimal) values of q_m corresponding to $m = \infty$. The method for our numerical estimations is based on replacing $W_{\text{per}}^{1,2}(Y)$ with a finite subspace in (3.1) and (3.2), and, the corresponding minima can thereby be found by standard FEM-programs. In this way we obtain lower and upper numerical estimates q_{num}^- and q_{num}^+ , respectively, for the actual effective conductivity q_1 .

In the table below we have listed q_∞/k_1 , $\bar{q}_1/k_1$, where

$$\bar{q}_1 = 1/2 (q_{\text{num}}^- + q_{\text{num}}^+),$$

the maximum relative numerical error estimate (in %)

$$r = 100 \frac{q_{\text{num}}^+ - q_{\text{num}}^-}{q_{\text{num}}^- + q_{\text{num}}^+},$$

and the number

$$d = 100 \frac{q_\infty - \bar{q}_1}{\bar{q}_1}$$

(which gives the maximum possible improvement in %) for some values of k_2/k_1 . In every case $v = 0.25$.

k_2/k_1	1.67	2, 5	5	10	100	∞
q_∞/k_1	0.88235	0.80645	0.71429	0.66038	0.60636	0.60000
$\bar{q}_1/k_1$	0.88220	0.80574	0.71186	0.65612	0.59941	0.59266
r	0.004	0.006	0.013	0.023	0.043	0.046
d	0.02	0.09	0.34	0.65	1.16	1.24

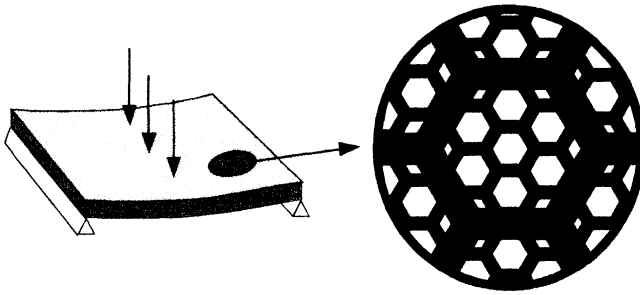


Figure 5.1: Sandwich-plate with regular reiterated honeycomb core.

5. Final comments

From the table in the previous section we observe that classical honeycomb-structure is nearly optimal (as expected). However, for large values of k_2/k_1 we see that it is possible to improve the effective conductivity with more than 1% by using reiterated honeycombs.

Related topics concerning numerical computations of effective properties can be found in e.g. Holmbom et al. [6] and Lukkassen et al. [11, 12].

The results obtained in Section 2 and 4 may be useful e.g. in optimum design of two-phase core materials in sandwich plates when the core is an unidirectional mixture of two isotropic materials with given shear moduli k_1 and k_2 (see Figure 5.1). In such constructions the stiffness is provided partly by the transverse shear strain resistance in the core material given by the effective transverse shear-module which usually is denoted G^* (see e.g. Meidell et al. [13]). In fact, it is possible to prove that G^* can be computed in the same way as the effective conductivity q given in (3.1).

By using the same ideas as those used in this paper it is possible to obtain similar results for anisotropic reiterated honeycombs (see Figure 5.2). We aim to discuss this further in a forthcoming paper.

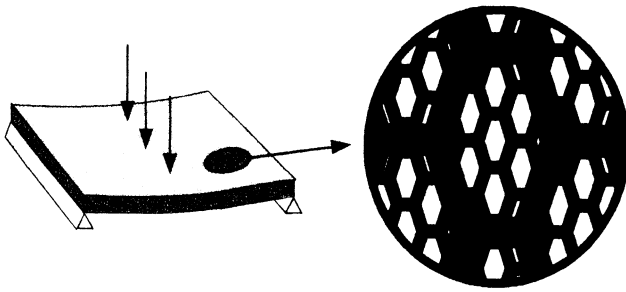


Figure 5.2: Sandwich-plate with anisotropic reiterated honeycomb core.

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