Shape/size optimization of truss structures using non-probabilistic description of uncertainty
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Abstract
In the present work an optimal solution for the well studied problem of truss shape/size optimization will be pursued when a certain level of uncertainty in the material constants is considered. This uncertain optimization will then give optimal structures which can be considered robust with respect to material constants variation. To this goal the convex set approach, and anti-optimization will be used to find the worst effect of the uncertainty on the structural response. The method consists in alternating between the main optimization problem and the maximization of constraints with respect to the variation of the uncertain variables. The main optimization is carried out using the Diagonal Quadratic Approximation of Zhang and Fleury. Numerical examples will be given using some simple classical cases of truss optimization.

1 Introduction
The aim of the present work is to study the effect of uncertainty on the solution of a shape/size optimization problem for truss structures. In design and optimization problems material constants, loading and structure geometry are usually considered as given data, but in real world, more often than not, assumed values do not correspond with actual ones, so that there may be differences between nominal and real geometry, materials may behave in a way different than the assumed one, constant loading may actually
vary during the structure lifetime etc. All this is usually accounted for by means of safety factors, which amplify load magnitude, or reduce material strength, leading in general to over-conservative structures.

As an alternative to safety factors one may try to describe the uncertain data and use this information during the optimization, which in general leads to better results in term of optimal design. Although probabilistic description is now-a-day very common, and very simple up to very sophisticated PDF can be used to describe uncertain parameters, convex model approach will be used here. Convex description simply fixes bounds for the uncertain variables instead of defining probability functions [1]. It then needs less information than probabilistic approach and is much easier to implement. During the optimization the uncertain domain is spanned when constraints are verified for a certain design in order to find the worst situation for each of them. This makes the solution robust against possible variation of the data within the assumed domain but creates a nested optimization which may be very expensive to solve.

To avoid this problem a two-step procedure was suggested in [7], where the search for the worst situation for the problem constraints, named anti-optimization [3], is done separately from the optimization and not within it. Final solution is obtained iterating between the two steps until convergence is obtained.

The method is here applied to the shape and size optimization of truss structures for minimum weight with displacements and stress constraints. Solution to this problem is found by solving a series of subproblems where all functions are approximated with quadratic polynomials as done by Zhang and Fleury [9]. Sensitivities used in both optimization and anti-optimization are evaluated analytically as first suggested by Svanberg [8].

2 Problem description

The general problem considered in the present work is the minimum weight optimization of a 3D truss structure with $NEL$ elements and $NNOD$, subject to constraints on node displacement and element stresses which can formulated as

$$\min_{A_i, x_{kl}} W(A, x)$$

s.t. $|u_j(A, x, E)| \leq \bar{u}_j \quad j = 1, \ldots, N_u$

$|\sigma_m(A, x, E)| \leq \bar{\sigma}_m \quad m = 1, \ldots, N_\sigma$

$A_i \in [A_{\text{min}}, A_{\text{max}}] \quad i = 1, \ldots, NEL$

$x_{kl} \in [x_{kl\text{min}}, x_{kl\text{max}}] \quad k = 1, 2, 3 \quad l = 1, \ldots, NNOD$

$E \in \Omega_E$
where $\mathbf{A}$ and $\mathbf{E}$ are the vectors of the element areas and Young’s moduli, $\mathbf{x}$ is the vector of coordinates, $u_j$ and $\sigma_m$ are the generic constrained nodal displacement and element stress, with $\tilde{u}_j$ and $\tilde{\sigma}_m$ the corresponding bounds, and where $\mathbf{A}^{\text{min/max}}$ are the allowed minimum and maximum value of the optimization variables. The uncertainty resides in the Young’s moduli vector $\mathbf{E}$ which is allowed to vary in the hypercube $\Omega_E$

$$\Omega_E = \{ \mathbf{E} \in \mathcal{R}^{N_{EL}} : E_i \in [E_i^0 - \delta_i, E_i^0 + \delta_i], i = 1, \ldots, N_{EL} \}$$

(2)

where the $\delta_i$’s define the accounted level of uncertainty.

### 2.1 Two-step method

As it can be seen from (1) the value of the constraint functions $u_j$ and $\sigma_m$ depends on the value of the vector of uncertain moduli $\mathbf{E}$ which may take any value in $\Omega_E$. Thus the first two constraints in (1) should be replaced with

$$u_{j,M}(\mathbf{A}, \mathbf{x}) \leq \tilde{u}_j \quad j = 1, \ldots, N_u$$

$$\sigma_{m,M}(\mathbf{A}, \mathbf{x}) \leq \tilde{\sigma}_m \quad m = 1, \ldots, N_{\sigma}$$

(3)

where

$$u_{j,M}(\mathbf{A}, \mathbf{x}) = \max_{\mathbf{E} \in \Omega_E} |u_j(\mathbf{A}, \mathbf{x}, \mathbf{E})|$$

(4)

$$\sigma_{m,M}(\mathbf{A}, \mathbf{x}) = \max_{\mathbf{E} \in \Omega_E} |\sigma_j(\mathbf{A}, \mathbf{x}, \mathbf{E})|$$

(5)

Problems (4) and (5), which are usually referred to as anti-optimization problems, should be solved for each trial design of the optimization problem. As previously mentioned this represent a nested optimization problem which, for large number of constraints and truss members may result very expensive to solve [4].

An alternative way of solving problem (1) was proposed in [7]. It consists in separating problems (4) and (5) from the main optimization problem and alternatively and iteratively solve the latter and the former until convergence occur, Fig. 1. Anti-optimization problems will give, for each constraint, an extremizing vector $\mathbf{E}^{(j)}$ and $\mathbf{E}^{(m)}$ which will be used during the optimization to verify the corresponding displacement and stress constraint. Solution to the optimization problem, i.e. optimal areas and node positions, in turn will be used in the anti-optimization to find a new set of $\mathbf{E}^{(j)}$ and $\mathbf{E}^{(m)}$’s and so on. The combination of anti-optimization(s) and optimization constitutes a cycle of the algorithm. The process, which usually starts with the anti-optimization, stops when there are no more changes in the design and in the vectors $\mathbf{E}^{(j/m)}$, or when these changes are smaller than a prescribed tolerance.
Start with initial \( A, x, E \)

**Anti-optimization**

find \( E_j \) to maximize constraints

**Optimization**

enter with vectors \( E_j \) and optimize for \( A \) and \( x \)

STOP

\[ \text{Y} \]

**Convergence ?**

Figure 1: Flow chart of the two step method

### 2.2 Application to truss structures

One of the big advantages of truss structures, besides their simplicity and their many applications, is the possibility of easily evaluate analytical sensitivities of the main structural properties. The key idea is to write the element stiffness matrix in a slightly modified form

\[
K_i = \begin{pmatrix}
\bar{K}_i & -\bar{K}_i \\
-\bar{K}_i & \bar{K}_i \\
\end{pmatrix}, \quad \bar{K}_i = \frac{E_i A_i}{\|r_i\|^2} \cdot r_i r_i^T
\]

where \( A_i \) and \( E_i \) are the \( i \)th element cross sectional area and Young’s modulus respectively and \( r_i \) is a vector that connects the two nodes of the element, such that \( \|r_i\| \) is the element length. It is worth to note that in this form \( K_i \) is already expressed in the global coordinate system, and is explicitly written in terms of the optimization variables. Doing so the analytical derivative of the element stiffness matrix with respect to both the area and the nodal position can be easily evaluated, and the the sensitivities of the global stiffness matrix can then be expressed as

\[
\frac{\partial K}{\partial A_i} = \sum_{j=1}^{\text{NEL}} \frac{\partial K_j}{\partial A_i}, \quad \frac{\partial K}{\partial x_{kl}} = \sum_{j=1}^{\text{NEL}} \frac{\partial K_j}{\partial x_{kl}}
\]

Sensitivities of \( K \) will then be used to evaluate the sensitivities of the constraint functions \( u_j \) and \( \sigma_m \) once these are expressed in terms of \( K \) and the corresponding pseudo-loads as was first proposed by Svanberg [8], and later improved by Zhang and Fleury [9]. In those works the shape/size optimization problem for a general 3D truss structure was solved using nominal data.

If uncertainty of the material must be taken into account, the same approach can be used to evaluate the derivatives of the constraints with respect to the uncertain problem data, which can be used to solve the anti-optimization problems. In particular the sensitivities of the displacement
\[ u_j \text{ with respect to the element Young's modulus } E_i \text{ is given as} \]
\[
\begin{align*}
\frac{\partial u_j}{\partial E_i} &= -K^{-1} \frac{\partial K}{\partial E_i} u \cdot \hat{e}_j, \quad j = 1, \ldots, N_u \\
\frac{\partial^2 u_j}{\partial E_i^2} &= -K^{-1} \left( \frac{\partial K}{\partial E_i} \frac{\partial u}{\partial E_i} \right) \cdot \hat{e}_j, \quad i = 1, \ldots, NEL
\end{align*}
\]
where the derivatives of the element stiffness matrices are given as in (7), and
\[
\frac{\partial K_i}{\partial E_j} = \begin{cases} A_i \|r_i\|^3 \cdot r_i r_i^T & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]
The sensitivities given by (8) are used to construct the second order approximation for the displacement which is used to evaluate the worsening vector of Young moduli \( E^{(j)} \). Note that only the diagonal terms of the Hessian matrix are evaluated, while cross terms are neglected. This is done to save computational effort and to maintain the same format used in the optimization algorithm which ensure separability of the variables.

In the present work only displacement constraints are anti-optimized, while stress constraints are verified with the nominal value of Young moduli \( E^0 \). The number of anti-optimization problems then equals the number of constrained displacements \( N_u \), and for each of them the first and second order sensitivities given by (8) are evaluated before starting the optimization by solving the structural problem. This may increase the computational cost by a considerable amount, depending on the number of constrained displacements, and eventually stresses, since the process must be repeated at each cycle of the optimization when sensitivities are updated. During the optimization then, the generic approximated constraint functions \( u_j \) are evaluated using the corresponding vector of Young's moduli \( E^{(j)} \) which has been found by the anti-optimization. This leads to a reduction of the feasible domain, forcing the optimization to search more conservative designs.

### 3 Numerical examples

A three bar truss problem with two displacement constraints and a five bar problem with one displacement constraint have been solved. Variables are the position of the free nodes and rod cross sections. The Young's modulus is allowed to deviate from its nominal value by a maximum of 1 percent. The anti-optimization is carried out using a program written by P. Spellucci [6] for solving non-linear optimization problems while the optimization is done with a code based on the convex linearization approach [5], with the diagonal quadratic approximation of Zhang and Fleury [9] for the objective function and the constraints. The two codes have been adapted and combined into a single program, solving the uncertain optimization. This
have been initially run with no uncertainties on classical truss problems and results were compared with those present in the literature and with those obtained with the second program using nominal data, showing a good agreement. In the following we will use the suffix uncertain for results obtained using uncertain material and nominal for those obtained with nominal value of data.

### 3.1 Three bar truss

The three bar truss in Fig. 2 has been first investigated. Variables of the problem are the areas of the three members and the coordinates of the top node. The Young modulus of the members is allowed to vary within ±1 percent from the nominal value $E_0 = 1000000$psi. The truss is loaded by loads $P_1 = 5000$lbs and $P_2 = 1000$lbs and stresses in all members are restricted to be less than 2000psi. The displacements of the top node are required to be smaller than 0.15in. Initial values of the variables and their corresponding ranges are shown in Table 1. Note than the value of the Young modulus and of the limit stress are similar but not the same as those of the aluminum material. These values have been selected so as to increase the magnitude of the sensitivities with respect to material constants. These were found to be very low (less than $10^{-6}$ and $10^{-12}$ first and second order respectively) and the anti-optimization program, based on modified gradient search, could not vary the given initial solution. Although this represent a limitation for the program, for the time being it was then decided to increase the modulus and verify the algorithm behavior.

To evaluate the goodness of results, besides the uncertain optimization and the nominal one, a third optimization has been carried out using the
lowerbound values of the Young modulus with no uncertainty, as if a safety
factor was used. Results of the three optimization are shown in Table 1
where $E^0$ indicates the nominal optimum design and $E^L$ indicates the opti-
mal design of the third optimization. First thing to notice is that all designs
are similar in terms of node position and areas: $A_1$, $A_3$, and $\sigma_3$ are at the
bound values and $A_1$ and $\sigma_3$ are only slightly different. Moreover for all
designs, displacement $u_2$ and stress $\sigma_3$ are binding, and the final weight is
also more or less attained on the same value, slightly more for the uncertain
optimum, as one would expect, and slightly less for the nominal one. The
third design is also better than the uncertain one, so why should one use the
anti-optimization? The reason lies in the fact that the while the constraints
violation of the nominal optimum due to a reduction of the $E_i$'s towards
their lower bound, may not be a surprise, the same thing happens with the
third design when the triplets of $E_i$ that maximizes the first displacements
in the uncertain design replaces the constant lower bound $E^L$ taking the
displacement $u_2$ beyond the 0.15in limit by 0.7 percent ($u_2 = 0.15105$in).
This shows that the worst-case is not necessarily the one corresponding to
the minimum value of the Young's modulus for all truss members, see also
Elishakoff [4], as it would happen with the application of safety factors to

|      | Initial Config. | Anti-Optim. $w_2 - u_2$ | Optim. $E^0$ | Optim. $E^L$
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<td>0.1500</td>
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Table 2: Initial and optimal designs for 5-BAR TRUSS problem

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<th>Optim. E°</th>
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</tr>
<tr>
<td>Iter.</td>
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Figure 3: Five-bar truss, initial (left) and final configurations

the uncertain parameters. One may say that 0.7 percent is a very low variation, but recall that the Young modulus was allowed to vary by almost the same amount, 1 percent, that is to say, with greater uncertainty one may obtain greater violation. It is finally interesting to note that the number of iteration to convergence, is comparable to the standard optimization.

3.2 Five bar truss

A five bar truss problem has been investigated. As for the three bar problem the uncertain solution have been compared with the nominal one to evaluate the variation in results corresponding to the accounted uncertainty. The structure, shown in Fig. 3, has five sizing variables, and four configuration variables. The symmetry of the problem has not been enforced and coordinates of nodes 2 and 4 are let independent from one another as well as areas of members 1 and 3, and members 4 and 5. This will increase the size of the problem. Initial configuration and truss data is displayed in Table 2 together with the solution of both optimization are shown in Table 2. Nominal solution is better than uncertain one but the latter become infeasible if the worsening vector of Young moduli is used to evaluate displacements. In this case the worsening vector of moduli is attained to the lower bound value, and so the optimal solution coincide with the one obtained using uniformly reduced $E_i$'s.

4 Conclusions and future works

The two step method and the Diagonal Quadratic Approximation using analytical sensitivities have been successfully used to solve uncertain shape/size optimization for truss structures. The method proved to be very efficient, on the case tested, and showed that even small allowed uncertainties in the material constants can lead to similar violation of the constraints not only for the optimal nominal solution, but also to that obtained by reducing the material property using a safety factor. Larger trusses, and stress constraints are next in the working schedule, which includes optimization of dynamic properties of the truss.
6 References


