Optimal multiple orientation synthesis of a planar parallel manipulator performed for different prescribed output workspaces

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Abstract

The optimal multiple orientation synthesis methodology presented in this paper is aimed at determining designs of a three degrees of freedom (3-dof) planar parallel manipulator, so that the manipulator reaches, with optimal conditioning, all points in a prescribed output workspace for multiple specified constant (fixed) orientations of the manipulator platform. The optimum conditioning with respect to the manipulator design variables is obtained by minimizing the condition number over the enclosed prescribed physical workspace for all the different prescribed discrete fixed orientations. The proposed optimization methodology produces convincing results, indicating it to be a stable and efficient numerical method for designing planar parallel manipulators.

Keywords: kinematics, parallel manipulators, workspace, optimal design.

1 Introduction

The optimal multiple orientation synthesis methodology presented here is aimed at determining designs of a three degree of freedom (3-dof) planar parallel manipulator, so that the manipulator reaches, with optimal conditioning, all points in a prescribed output workspace for multiple specified constant (fixed) orientations of the manipulator platform. In previous work the authors developed a constrained optimization methodology for the optimal synthesis of 3-dof planar manipulators for a prescribed output workspace with a single fixed orientation of the platform specified, and applied the procedure separately to the design for three different prescribed workspaces [1]. The method was subsequently extended in a white paper...
[2] and in a recent journal article [3] to apply to multiple orientations, and was demonstrated for a single prescribed output workspace. In this paper the work on the optimal synthesis for multiple orientations, is formally completed with a detailed presentation of the method, and the results of its application to three different prescribed output workspaces. In particular the optimum design with respect to the manipulator design variables, is obtained by minimizing the condition number over the enclosed prescribed physical workspace for three different fixed orientations, namely with the platform assuming orientations of $-5^\circ$, $0^\circ$ and $+5^\circ$. Such a design may be expected to fulfill, or nearly fulfill, the dexterity requirement of operating over the continuous orientation range of $-5^\circ$ to $+5^\circ$.

In the next two sections the geometry of the particular Gough-Stewart type 3-dof planar parallel manipulator of interest here is presented, and the kinematics and the determination of the condition number for this manipulator are discussed. Since the parallel manipulator inverse kinematics is easy to solve, it allows for the characterization of the performance of the 3-dof planar parallel manipulator, in terms of the inverse of the condition number of the manipulator Jacobian matrix. This is important since the accuracy of control of the manipulator is dependent on the condition number. The condition number therefore plays an important role in the optimal synthesis methodology presented in detail in the later sections of this paper.

## 2 The three-degree-of-freedom parallel manipulator

The manipulator considered in this paper is the 3-dof planar parallel mechanism shown in Figure 1. The manipulator consists of a platform of length $2r$ connected to a base by three linear actuators, which control the three output degrees of freedom of the platform. The actuators have leg lengths $l_1$, $l_2$ and $l_3$ and are joined to the base and platform by means of revolute joints identified by the letters $A − E$. It will be assumed that $y_C = y_D = y_E$. The coordinates of point $P$, the mid-point of the platform, are $(x_P, y_P)$ and the orientation of the platform is $\phi_P$. In more general terms, the actuator leg lengths are the input variables, i.e. $v = [l_1, l_2, l_3]^T \in \Re^3$. The global coordinates of the working point $P$ form the output coordinates, i.e. $u = [x_P, y_P]^T \in \Re^2$. The 3-dof manipulator may, in addition to positioning $P$ in the $x-y$ plane, be orientated at an angle $\phi$ by controlling the three leg lengths. It is evident that this manipulator thus has three degrees of freedom. The rotation angle of the platform is considered as an intermediate coordinate $w = \phi_P$. Thus, the generalized coordinates for this platform are therefore given by

$$ q = [u^T, v^T, w]^T = [x_P, y_P, l_1, l_2, l_3, \phi_P]^T \in \Re^6 \quad (1) $$

In the vicinity of an assembled configuration the input, output and intermediate coordinates satisfy the 3 independent kinematic constraint equations of the general form

$$ \Phi(q) = \Phi(u, v, w) = 0 \quad (2) $$
In general, factors imposed by the physical construction of the planar parallel manipulator, which limit the workspace, may be related to the input variables or a combination of input, output and intermediate variables. An example of former type for the planar parallel manipulator are leg length limits, and of the latter, limits on the angular displacement of the revolute joints connecting the legs to the ground and to the platform. These limiting factors are described by means of inequality constraints and may respectively take the general forms

\[ v_{\text{min}} \leq v \leq v_{\text{max}} \]  

(3)

\[ g_{\text{min}} \leq g(u, v, w) \leq g_{\text{max}} \]  

(4)

Limits on the platform orientation (intermediate coordinate) take one of two forms given by

\[ w_{\text{min}} \leq w \leq w_{\text{max}} \]  

(5)

or \[ w = w_{\text{fix}} \]  

(6)

where \( w_{\text{fix}} \) is a prescribed fixed scalar quantity.

The above general definitions are necessary in order to facilitate the mathematical description of kinematics and workspaces of the 3-dof planar parallel manipulator. In this study mixed constraints, represented by 4, are not taken into consideration.
3 The kinematics and condition number of the manipulator

In general, the parallel manipulator inverse kinematics are easy to solve. For the manipulator under consideration, the three leg lengths are given by

\[
\begin{align*}
l_1^2 &= (x_P - r \cos \phi_P - x_C)^2 + (y_P - r \sin \phi_P - y_C)^2 \\
l_2^2 &= (x_P - r \cos \phi_P - x_D)^2 + (y_P - r \sin \phi_P - y_D)^2 \\
l_3^2 &= (x_P + r \cos \phi_P - x_E)^2 + (y_P + r \sin \phi_P - y_E)^2
\end{align*}
\] (7)

Writing in the standard form of the kinematic constraint equations (2) and using the general coordinates definitions from the previous section, (7) become

\[
\begin{bmatrix}
\begin{align*}
v_1^2 &= (u_1 - r \cos w - x_C)^2 - (u_2 - r \sin w - y_C)^2 \\
v_2^2 &= (u_1 - r \cos w - x_D)^2 - (u_2 - r \sin w - y_D)^2 \\
v_3^2 &= (u_1 + r \cos w - x_E)^2 - (u_2 + r \sin w - y_E)^2
\end{align*}
\end{bmatrix}
= 0
\] (8)

The explicit expressions for \( v \) in terms of \( u \) and \( w \), \( v = v(u, w) \), may be determined from (7), allowing constraints (3) to be written as follows:

\[
v_{\min} \leq v(u, w) \leq v_{\max}
\] (9)

where \( v_{\min} = [l_{1\min}, l_{2\min}, l_{3\min}]^\top \) and \( v_{\max} = [l_{1\max}, l_{2\max}, l_{3\max}]^\top \).

As in [1], the specific performance measure used here to characterize the performance of the 3-dof planar parallel manipulator is the inverse of the condition number of the Jacobian matrix of the manipulator. The accuracy of control of the manipulator is dependent on the condition number, denoted here by \( \kappa \). Since \( \kappa \) tends to infinity as the manipulator approaches a singular position, maximizing the inverse condition number, \( \kappa^{-1} \), also ensures that the manipulator remains far away from singular positions. From (2), an inverse transformation relating the input, output and intermediate velocities can be determined:

\[
J_\theta \dot{\theta} = -J_\nu \dot{\nu}
\] (10)

where \( \theta = [u^\top, w]^\top \), and \( J_\theta \) and \( J_\nu \) are the respective constraint Jacobian matrices containing the partial derivatives of the 3 kinematic constraints (2) with respect to the variables \( \theta \) and \( \nu \). Equation (10) can be rewritten as

\[
J \dot{\theta} = \dot{\nu}
\] (11)

where \( J = -J_\nu^{-1}J_\theta \).

One point now arises due to the platform’s orientational ability. The Jacobian of the manipulator contains entries related to both positional and rotational abilities of the platform. The condition number will thus inherently contain a mix of these terms. It is important to normalize the positional terms of the Jacobian matrix...
so that positional and rotational abilities are equally represented by the condition number. Pittens and Podhorodeski [4] and Stoughton and Arai [5] note this occurrence and suggest that the best approach is to normalize the positional terms of the Jacobian with respect to the platform radius $r$, a suggestion which is adopted here.

In explicit terms, differentiation, with respect to time, of the kinematic constraints (2), and writing in the form (10) yields

$$
\begin{bmatrix}
  x_{AC} & y_{AC} \\
  x_{AD} & y_{AD} \\
  x_{BE} & y_{BE}
\end{bmatrix}
\begin{bmatrix}
  x_{AC} & y_{AC} & rx_{AC} \sin w - ry_{AC} \cos w \\
  x_{AD} & y_{AD} & rx_{AD} \sin w - ry_{AD} \cos w \\
  -rx_{BE} \sin w + ry_{BE} \cos w & y_{BE}
\end{bmatrix}
\begin{bmatrix}
  \dot{u}_1 \\
  \dot{u}_2 \\
  \dot{w}
\end{bmatrix}
= 
\begin{bmatrix}
  v_1 \\
  v_2 \\
  v_3
\end{bmatrix}
\begin{bmatrix}
  \dot{v}_1 \\
  \dot{v}_2 \\
  \dot{v}_3
\end{bmatrix}
$$

(12)

where the notation $x_{AB} = x_A - x_B$ is used, and $x_A = u_1 - r \cos w$, $y_A = u_2 - r \sin w$, $x_B = u_1 + r \cos w$ and $y_B = u_2 + r \sin w$.

The Jacobian $J$ of the 3-dof planar manipulator, as defined by (11), is thus given by

$$
J =
\begin{bmatrix}
  x_{AC}/rv_1 & y_{AC}/rv_1 & (rx_{AC} \sin w - ry_{AC} \cos w)/v_1 \\
  x_{AD}/rv_2 & y_{AD}/rv_2 & (rx_{AD} \sin w - ry_{AD} \cos w)/v_2 \\
  x_{BE}/rv_3 & y_{BE}/rv_3 & (-rx_{BE} \sin w + ry_{BE} \cos w)/v_3
\end{bmatrix}
$$

(13)

Note the normalization of the positional terms in the first two columns by the platform radius $r$. The condition number $\kappa$ of the $n \times n$ Jacobian $J$ is defined as

$$
\kappa = \|J\|\|J^{-1}\|
$$

(14)

where $\|\cdot\|$ denotes any norm of its matrix argument. The norm adopted here is the same as that used by Gosselin and Angeles [6], namely

$$
\|J\| = \sqrt{\text{trace}(JWJ^T)}
$$

(15)

where $W$ is defined as $n^{-1}$ multiplied by the $n \times n$ identity matrix. The lower the condition number, the better the behavior of the manipulator, with the lowest possible value of $\kappa$ being unity. The value of $\kappa^{-1}$, the inverse of the condition number, thus lies between 0 and 1, and is preferably used as it is bounded and better conditioned than the condition number itself.

4 Determination of workspaces

4.1 Dextrous and constant orientation workspace definition

The general dextrous requirement for the manipulator at a point $u$ is that all orientations in the range

$$
\phi^{\min} \leq w \leq \phi^{\max}
$$

(16)
can be attained by the manipulator. The dextrous workspace $W_D[\phi_{\min}, \phi_{\max}]$ of the planar manipulator is thus defined as:

$$W_D = \{u \in \mathbb{R}^2 : \Phi(u, v, w) = 0, \text{ with } v \text{ satisfying (3)}$$

and $g(u, v, w)$ satisfying (4) for all $w \in [\phi_{\min}, \phi_{\max}]\}$$

The boundary $\partial W_D$ of the dextrous workspace can thus be defined as:

$$\partial W_D = \{u \in \mathbb{R}^2 : u \in W_D \text{ and } \exists \text{ an } s \in R^2 \text{ such that for (18)}$$

$$u' = u + \lambda s, \lambda \in \mathbb{R} \text{ arbitrarily small and either positive or negative, no } v \text{ exists that satisfies } \Phi(u', v, w) = 0;$$

as well as inequalities (3) and (4) for all $w \in [\phi_{\min}, \phi_{\max}]\}$$

The above definition of course includes that of the workspace for which the orientation, is constant, i.e. fixed at $\phi_P = \phi_{\text{fix}}$, which is the case of interest here. In this case $\phi_{\min} = \phi_{\max} = \phi_{\text{fix}}$. The corresponding constant orientation workspace is denoted by $W_C[\phi_{\text{fix}}]$.

4.2 Computation of workspaces

Much work has been done on the numerical determination of the workspaces of parallel manipulators via an optimization approach. The earliest work is that of [7, 8, 9] in which the workspace determination of both planar and spatial parallel manipulators were determined. With regard to the particular 3-dof manipulator of interest here, the so-called chord method for determining fixed orientation workspaces (Hay and Snyman [10]), has recently been refined [3], and has also been extended to the determination of more general dextrous workspaces (Hay and Snyman [11]). The validity and reliability the chord method, an alternative to more established geometrical approaches (Merlet [12]), is now generally accepted. In what follows the chord method was used to determine fixed orientation workspaces where required. Of course any other valid method for workspace determination could also have been used, without affecting the generality of the proposed methodology for optimal dexterity design.

5 Optimal synthesis for a single prescribed constant orientation workspace

It is assumed the actuators have been chosen, and thus that the maximum and minimum leg lengths are predetermined. The remaining five design variables for the problem are thus (see Figure 1):

$$d = [x_C, y_C, x_D, x_E, r]^T$$

Before tackling the more challenging problem of the optimal synthesis for multiple prescribed orientations prescribed continuous dexterity requirement, it is necessary
As described in [11], the prescribed workspace is defined by polar coordinates \((\beta_p, r_p)\) centered on a local coordinate system \(x' - y'\) at \(O'\). The boundary of the constant workspace \(W^C_{\phi_{\text{fix}}}\) associated with design \(d\) is represented in a similar manner (see Figure 2). The chord method [10] is used to generate points \(b_{ci}\), with corresponding polar coordinates \((\beta_{ci}, r_{ci})\), on the constant orientation workspace boundary.

Dropping the \([\phi_{\text{fix}}]\), which is implicit when referring to constant orientation workspaces \(W^C\) for the rest of this section, the part of the prescribed workspace \(W^C_p\) not intersecting calculated workspace \(W^C_c\) is denoted \(\delta W^C_p\), and the part of workspace \(W^C_c\) not intersecting \(W^C_p\) is denoted \(\delta W^C_c\). The calculation of approximations to the areas \(\delta W^C_p\) and \(\delta W^C_c\) is performed using the numerical scheme described in [10].
Single orientation synthesis is thus achieved by solution of the following optimization problem:

$$\max_{\mathbf{d}} f(\mathbf{d}) = \max_{\mathbf{d}} \left\{ \min_{\mathbf{u} \in W^C_p[\phi]} \kappa^{-1}(\mathbf{d}, \mathbf{u}) \right\}$$

subject to the inequality constraint

$$g(\mathbf{d}) \leq 0$$

where for a design \(\mathbf{d}\), \(\kappa\) may be measured at any point \(\mathbf{u}\) within the prescribed workspace \(W^C_p\), and the inequality constraint function \(g(\mathbf{d})\) relates to the mismatch between \(W^C_p\) and \(W^C_c\). The displacement vector between the prescribed workspace boundary and calculated workspace boundary (see again Figure 2), measured along a ray emanating from \(O'\) at angle \(\beta^c_i\) is denoted by \(r^i e^i\), where \(e^i\) is a unit outward vector at angle \(\beta^p_i\). If \(r_{\text{min}} = \min_i \{|r^i|, i = 1, 2, \ldots, n_{bc}\}\), then setting \(r = r_{\text{min}}\), the constraint function as follows:

$$g(\mathbf{d}) = \begin{cases} \delta W^C_p & \text{if } \delta W^C_p > 0 \\ -r^2 & \text{if } \delta W^C_p = 0 \end{cases}$$

Therefore, the solution to optimization problem (20) seeks a design which not only gives a workspace that economically and fully encloses the prescribed workspace, but in such a manner as to improve the single worst point with respect to the chosen performance measure, \(\kappa^{-1}\), within the prescribed workspace, \(W^C_p\).

The question of how to determine the smallest value of \(\kappa^{-1}\) over the set \(\mathbf{u} \in W^C_p\) arises. Previous theoretical arguments [1] have indicated, and numerical experiments have borne out the fact that, the extreme values of \(\kappa\) always lie on the boundary \(\partial W^C_p\) of the prescribed workspace. The minimum value of the inverse condition number \(\kappa^{-1}\) can thus be approximated by calculating \(\kappa^{-1}\) at points \(b^i_p, i = 1, \ldots, n_{bc}\) (again see Figure 2) simultaneously to the determination of the boundary points \(b^i_c, i = 1, \ldots, n_{bc}\) by the chord method. The overall minimum of the \(\kappa^{-1}\) values at these candidate points may then easily be determined.

The method outlined above for single orientation synthesis, has successfully been applied to the optimal synthesis of the 3-dof planar parallel manipulator for prescribed workspaces, in [3] for a single output workspace, and in Hay [13] for three different prescribed output workspaces.

### 6 Optimization for multiple prescribed constant orientation workspaces

The multiple orientation (MO) synthesis problem is now tackled, namely that of determining a manipulator design that reaches a prescribed output workspace, with optimal conditioning, for multiple prescribed constant orientation workspaces.
6.1 Optimization formulation

In the methodology proposed here, the minimization over \( u = [x, y]^\top \) in (20) is carried out, not only for a single prescribed value of \( \phi_P \), but over a multiple \( m_{sl} \) fixed values of \( \phi_P \). Thus optimization problem (20), modified to allow for optimization over \( (m_{sl}) \) values of \( w = \phi_P \), becomes

\[
\max_d f(d) = \max_d \left\{ \min_{u \in W_P^C[\phi_i]} \kappa^{-1}(d, u) \right\}
\]

subject to the inequality constraint

\[
g(d) \leq 0
\]

The inequality constraint function is defined as follows:

\[
g(d) = \begin{cases} 
S = \sum_{i=1}^{m_{sl}} \delta W_P^C[\phi_i], & i = 1, \ldots, m_{sl} \quad \text{if } S > 0 \\
-\rho^2 & \text{if } S = 0
\end{cases}
\]  

(23)

This formulation is identical to that given in (20), except that there are now \( (m_{sl}) \) relevant fixed orientation workspaces to be considered in (22) and (23). The minimum value of the condition number is determined using the same approach as given in Section 5. Once again it is assumed that the minimum value of \( \kappa^{-1} \) occurs on the boundaries of the prescribed fixed orientation workspaces. The results reported below indicate this assumption to be valid.

6.2 Numerical results

Actuator limits were again chosen as \( l_{i_{\min}} = \sqrt{2}, \quad l_{i_{\max}} = 2 \), \( i = 1, 2 \) and \( l_{3_{\min}} = 1, \quad l_{3_{\max}} = \sqrt{3} \). Three fixed orientations, i.e. \( m_{sl} = 3 \), was selected with corresponding orientations \( \phi_P = -5^\circ, 0^\circ, +5^\circ \). The prescribed workspaces P1-P3, corresponding to those specified in the Appendix, are shown in Figure 3. The manipulator workspaces corresponding to an initial design vector \( d_0 = [-0.75, 0, 0.75, 1.5, 0.75]^\top \) for the various constant orientations, as well as the corresponding inverse condition number contours are shown in the same figure.

Figure 3 clearly shows that the initial design \( d_0 \) is infeasible because for each specified orientation the prescribed workspace is not contained in the reachable workspace. The optimization problem embodied in (22) was solved using the Dynamic-Q algorithm [14]. The Dynamic-Q optimization algorithm move limit used was \( \rho = 0.1 \) and the chord length for calculating the workspace was \( d = 0.02 \). Convergence tolerances used for Dynamic-Q were \( \varepsilon_x = 10^{-4} \) and \( \varepsilon_f = 10^{-5} \) and a finite difference of \( \Gamma = 10^{-6} \) was used for calculating function gradients.

The workspaces, and \( \kappa^{-1} \) contours corresponding to the optimal designs for prescribed workspaces P1-P3 are shown in Figures 4 to 6. Table 1 summarizes the number of gradient evaluations \( N^g \) required for convergence, the initial \( f(d^0) \) and
Table 1: MO synthesis solutions.

<table>
<thead>
<tr>
<th>N^g</th>
<th>f(d^0)</th>
<th>f^*</th>
<th>g^*</th>
<th>d^*</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>33</td>
<td>0.677</td>
<td>0.901</td>
<td>0.2E − 5</td>
</tr>
<tr>
<td>P2</td>
<td>25</td>
<td>0.681</td>
<td>0.917</td>
<td>0.1E − 4</td>
</tr>
<tr>
<td>P3</td>
<td>44</td>
<td>0.666</td>
<td>0.915</td>
<td>0.1E − 4</td>
</tr>
</tbody>
</table>

Figure 3: MO synthesis: prescribed workspaces P1-P3 and manipulator workspace and κ−1 contours corresponding to the starting design.

Figure 4: MO synthesis: manipulator workspace and κ−1 contours corresponding to the optimal design for prescribed workspace P1.

The final f^* objective function values, inequality constraint function value at convergence g^* and optimal design d^* for each prescribed workspace. Figure 7 shows the convergence histories for the various optimization runs.

7 Conclusion

The proposed optimization methodology produces convincing results, indicating it to be a stable and efficient numerical method for designing planar parallel manipulators. The synthesis methodology yields manipulator dimensions that ensure, that
for all the specified fixed orientations, that each point within the prescribed output workspace can be reached with optimal conditioning.

The methodology developed here also points to a strategy for optimal synthesis for a dextrous workspace. If the optimum design, with respect to the manipulator design variables, is obtained by minimizing the condition number over the
Table 2: Parameters specifying the prescribed workspace for O synthesis.

<table>
<thead>
<tr>
<th>$[\beta_p \min, \beta_p \max)$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-\pi/4, \pi/4)$</td>
<td>$-D \cos(\beta_p)$</td>
</tr>
<tr>
<td>$[\pi/4, 3\pi/4)$</td>
<td>$-D \sin(\beta_p)$</td>
</tr>
<tr>
<td>$[3\pi/4, 5\pi/4)$</td>
<td>$+D \cos(\beta_p)$</td>
</tr>
<tr>
<td>$[5\pi/4, 7\pi/4)$</td>
<td>$+D \sin(\beta_p)$</td>
</tr>
</tbody>
</table>

Table 3: Parameters specifying prescribed workspace P3.

<table>
<thead>
<tr>
<th>$[\beta_p \min, \beta_p \max)$</th>
<th>$dx$</th>
<th>$dy$</th>
<th>$R$</th>
<th>$\pm$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0, \pi/2)$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.2857</td>
<td>+</td>
</tr>
<tr>
<td>$[\pi/2, \pi)$</td>
<td>-0.2143</td>
<td>0.0000</td>
<td>0.3571</td>
<td>+</td>
</tr>
<tr>
<td>$[\pi, 1.444\pi)$</td>
<td>0.3929</td>
<td>0.3124</td>
<td>0.4072</td>
<td>-</td>
</tr>
<tr>
<td>$[1.444\pi, 1.555\pi)$</td>
<td>-0.01000</td>
<td>0.1238</td>
<td>0.04172</td>
<td>-</td>
</tr>
<tr>
<td>$[1.555\pi, 2\pi)$</td>
<td>-0.3304</td>
<td>0.3571</td>
<td>0.3599</td>
<td>-</td>
</tr>
</tbody>
</table>

enclosed prescribed physical workspace for a large number ($m^{sl}$ large) of discrete fixed orientations of the platform between $\phi_P = \phi^\min$ and $\phi_P = \phi^\max$, then such a design may be expected to fulfill, or nearly fulfill the dexterity requirement of operating over the continuous orientation range of $\phi^\min$ to $\phi^\max$. This is similar to that used by Boudreau and Gosselin [15] in an unconstrained case.

The Dynamic-Q optimization algorithm [14] used in the synthesis methodology exhibits proved to be reliable and efficiency in solving the associated optimization problem. Also, the chord method used for computing the workspaces used in the optimization formulation, again proved to be highly reliable and economical.

**Appendix: Prescribed manipulator output workspaces**

Three different manipulator output workspaces, denoted P1-P3, are prescribed. P1 is the workspace considered by Gosselin and Guillot [16]. The boundary of the prescribed workspace is defined in polar coordinates relative to a local coordinate system centered at $O' = [0, 1.5]^\top$:

$$r_p(\beta_p) = \alpha + \sqrt{D^2 \cos^2(\beta_p) + R^2 - D^2}$$

where $R = 0.5735$ and $D = 0.3755$ and the expression for $\alpha$ for various intervals of $\beta_p$ is given in Table 2.

P2 is an ellipse centered at $O' = [0, 1.5]^\top$ with $x$ and $y$ half axis lengths $a = 0.35$ and $b = 0.20$. P3 is a non-convex, non-symmetrical workspace centered at
\( O' = [0, 1.5] \top \) and defined in polar coordinates by

\[
 r_p(\beta_p) = -b \pm \sqrt{b^2 - dx^2 - dy^2 + R^2}
\]

where \( b = dx \cos \beta_p + dy \sin \beta_p \)

and where the parameters \( dx, dy, R \) and the sign before the square root are, for various angular intervals \([\beta_{p\text{min}}, \beta_{p\text{max}}])\), as given in Table 3. The boundary thus consists of five smooth arcs.

References


