Shape optimization: an analytical approach

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Abstract

The scientific field of biomechanics deals with the statics and dynamics of biological load carriers and the transfer of biological optimization strategies to technical applications. By observation of biological load carriers in nature a fundamental design rule could be found called “the axiom of uniform stress”, which basically says that in the temporal average with a certain load there are neither high local stresses nor underloaded areas in a biological load carrier. Through load-controlled growth the load carrier develops material in areas with high stress as in areas with low stress there is no growth and hence no material. From this a shape optimization is obtained which is able to prevent fatigue failure. The adaptive growth of biological load carriers is simulated by the CAO-Method (Computer Aided Optimization) and the FEM. With the help of the CAO-Method a parameter study for the optimization of shoulder fillets was made from which a geometrical characterization of the optimized fillet shapes was derived. From that new ways for shape optimization were developed which offer design rules using alternative fillet shapes and a better understanding of the effects caused by stress concentrations was obtained. Due to this knowledge a new and efficient optimization method was developed which offers a fast and simple way to construct optimized fillet shapes. This method can be expressed analytically, so thus does not require much calculating power and was verified with several examples.

Keywords: shape optimization, biomechanics, stress concentration.

1 Introduction

Notch stresses cause many failures in engineering components, because of a stress concentration effect located at the notch. In order to get a high fatigue resistance of the component, it is essential to lower these stresses. This is
achieved by changing the shape of the component in the area, where the stress concentration is located. An important design rule for these shape modifications comes from biological load carriers. Trees and bones, e.g., try to obtain a uniform stress distribution on their surface by adaptive growth. They have an increase of material, where the stress concentration is to be found in order to reduce the stress peak. This design rule is called the axiom of uniform stress. Using the finite element method in combination with the axiom of uniform stress, an optimization method has been developed at the Research Centre Karlsruhe, which simulates biological growth on engineering components.

Based on calculations with the CAO-Method (Computer Aided Optimization Method) a deeper understanding of notch stresses helped to develop an even more efficient way of optimizing fillet shapes. This new optimization method comes with an analytical expression for creating optimized fillet shapes.

2 Computer Aided Optimization method (CAO)

The CAO (Computer Aided Optimization) method simulates by a FE-calculation a fictitious thermal expansion dependent on previously calculated stresses. Therefore, high stresses lead to a great thermal expansion and vice versa. The stress peaks will be reduced and finally a homogeneous state of stresses on the surface is achieved. First a finite element structure of the component is generated. Then a FE-analysis of the model is calculated. The computed stresses are formally set equal to a fictive temperature distribution. Then the modulus of elasticity is set at 1/400 of its initial value in the growth layer. The previous mechanical load is set at zero. Only the soft surface layer of the component will have a thermal expansion factor larger than zero. The result is a soft upper layer which is hot at the over-loaded zones and rather cold in the less loaded zones. After that a thermal FE-analysis is computed where the zones with the previously highest loads now have the highest temperature and expand most strongly. The component starts to grow in the highly loaded zones and thus modifies the geometry of the component. The procedure is done iteratively until a uniform stress state on the surface is achieved [1].

3 Shape optimization

3.1 Shoulder fillets

One of the most frequently encountered type of stress concentration in machine design is the shoulder fillet. The various diameters of a shaft, spindle or rotor are connected often by shoulders with circular fillets to avoid sharp corners. These circular fillets represent a more or less good way to design a shoulder fillet of a shaft, concerning the stress concentration at these fillets. Therefore, the CAO method was used to generate shoulder fillet shapes for a shaft with different lengths, which result in different stress reductions. The diameter ratio of a shaft was chosen to 3.0 as an example for a parameter study where several diameter ratios and 3 different load cases are included.
The shaft was subjected to pure tension. The result of the optimization is shown in fig. 1, where several shoulder fillets for predefined stress reductions have been generated. The axial design space of the fillet is described with the ratio a/d, where a is the absolute axial length of the fillet and d is the small diameter of the shaft. The circular fillet has an axial and radial design space of 0.08, while the lengths of the CAO-fillets are continuously increasing with the stress reduction, that comes with the fillet.

The related stress diagram shows the ratio of the von Mises stress along the contour s and the nominal stress $\sigma_{\text{Mises}}/\sigma_{\text{ref}}$. One can see the stress peak generated by the circular fillet. This leads to a stress concentration of 1.97 located in the fillet. The CAO-fillet with an axial design space a/d of 0.069 reduces the stress in the fillet to a stress concentration of 1.8, which means a stress reduction of 9%. The modified fillet shape homogenizes the stress peak of the circular fillet, so that there is a plateau in the stress distribution along the contour at 1.8. This plateau is getting longer with preceding optimization as one can see in the diagram on the right hand side of Figure 1. The CAO-fillet with an axial design space of 0.102, obtains a stress concentration of 1.6 and therefore a stress reduction of 19%. Fillets with axial design spaces of 0.148 and 0.258 result in stress concentrations of 1.4 and 1.2 and stress reductions of 29% and 39%, respectively.

![Figure 1: CAO-optimization of shoulder fillets for a shaft under tension load.](image)

**3.2 Geometrical characterization of CAO-fillets**

From the parameter study on optimizing shoulder fillets in shafts a geometrical characterization of shape-optimized fillets can be derived [2]. Fig. 2 shows optimal shoulder fillets with a stress concentration of 1.2 for different diameter ratios D/d and different load cases.

One can see a great similarity between fillet shapes for shafts with high diameter ratios, which decreases for ratios smaller than 1.4. The first diagram shows the results for a shaft under tension load, where the entry angles of the fillets into the shoulder are relatively small compared to the entry angles of fillets for a shaft under bending load, which are shown in the second diagram.
fillets for a shaft under torsion load are shown in the third diagram, where the highest entry angles of fillets can be stated.

![Figure 2: CAO-fillets for shafts under various load cases and diameter ratios.](image)

Another characteristic of an optimized fillet is its axial and radial design space a/d and b/d. It can be described as the design space ratio a/b depending on the axial design space a/d. Fig. 3 shows this characteristic for shafts and three different load cases. All load cases have in common that for a constant diameter ratio there is always a linear relation between the design space ratio and the axial design space for all fillets with different stress concentrations for that shaft. Thereby, the slope of the design space ratio is small for high diameter ratios and is increasing with decreasing diameter ratios. For a shaft under tension load the design space and the design space ratios are higher than for a shaft under bending load, while a shaft under torsion load has the smallest values for these two properties.

4 Analytical approach

4.1 Notices on notch stresses

4.1.1 Stress concentration depending on force flow deflection

The curvature of a notch and thus the force flow deflection around this notch has a strong effect on the stress concentration in this notch. To examine the influence
of force flow deflection only on the stress concentration, a shaft under tension load was modelled with a varying shoulder angle. The diameter ratio is 1.63, the shoulder angles range from 0° to 150° and the fillet radius is for all shoulder angles constant with R/d=0.02. Fig 4. shows the stress concentration depending on the shoulder angle. One can see that with an increasing shoulder angle there is a decreasing stress concentration in the fillet. From 0° to 90° there is only a small change in the stress concentration. This is about to change for angles higher than 90° where the decrease of the stress concentration is very high.

![Graph showing stress concentration vs shoulder angle](image)

Figure 3: CAO-design spaces for shafts.

4.1.2 Tapered fillets

A relatively weak force flow deflection would result in a high shoulder angle and thus in a large design space. In order to lower the design space, but keep a weak force flow deflection, a tapered fillet with varying taper angles was investigated concerning the stress concentration at the two circular notches in this fillet. Again a shaft under tension load was modelled with a diameter ratio of 1.63. The taper angles range from 14.6° to 59.5° while the radius of the circular notches keeps constant at R/d=0.02. For very small taper angles the force flow deflection, and thus the stress concentration, is very low at the notch on the small diameter of the shaft and very high at the notch of the shoulder entry. For higher taper angles it turns to be the other way around, that the shoulder entry features the smaller stress concentration than the notch on the small diameter. But for one
taper angle the stress concentration at both notches is equal. A constant distribution of stress concentrations.

Figure 4: Stress concentration depending on the shoulder angle of a shaft.

Figure 5: Stress concentration depending on the taper angle of a shaft.

4.1.3 A multi-linearized CAO-fillet

The tapered fillet is a fillet with one segment and two notches. There is a certain taper angle where the stress concentrations of the notches are equal. In order to further lower the stress concentrations, a fillet with five segments was modelled. In this case the optimum angles of the segments are hard to find. Because of that the CAO-fillet act as an example for the multi-linearized fillet. The result is a nearly constant distribution of stress concentrations, which are also smaller than the ones of the tapered fillet (fig. 6).
4.2 Method

In order to provide a way of getting the optimum angles for a multi-linearized fillet without using the CAO-fillet as an example, an analytical approach was developed [3]. This approach is based on the relation between notch stresses and the non-linear bending stress distribution of a curved bar [4,5]. The result is a formulation which has its optimum when the increase of the transverse tension stresses due to the increase of curvature is cancelled out by the decrease of the nominal stresses due to the increase of the diameter. In other words: the higher the notch stresses (increasing curvature) get, the more material (increasing diameter) is required in order to keep a constant stress distribution along the contour of the fillet.

The angles between the segments of an optimal fillet (fig. 7) are expressed in the following equation:
\[ \alpha_{i+1} = 2 \arcsin \left( \frac{D_{i+1}}{D_i} \cos \left( \sum_{k=0}^{i-1} \alpha_k \right) \left( 1 + 2 \sin \left( \frac{\alpha_i}{2} \right) \right) - \cos \left( \sum_{k=0}^{i} \alpha_k \right) \right) \]
\[ 2 \cos \left( \sum_{k=0}^{i} \alpha_k \right) \]

with \( D_{i+1} = D_i + 2s \sin \left( \sum_{k=0}^{i} \alpha_k \right) \),

\( \alpha_0 = 0^\circ \) and an empirical initial angle, e.g. \( \alpha_1 = 3^\circ \)

For many applications an angle \( \alpha_L = 45^\circ \) is an appropriate value.

### 4.3 Examples

The analytical formulation provides a fillet contour in linear pieces, which is smoothed by a fitting curve. This curve can be enlarged or reduced in similar shape and thus adapted to the available design space. Some examples of the analytical fillet in comparison with the CAO-fillet and the conventional circular fillet are given below.

#### Figure 8: Shaft under tension load.

#### Figure 9: Metric screw under tension load.
5 Experiments

Shafts with analytical fillets, CAO-fillets and circular fillets were tested in a repetitive bending test and the results are given in fig. 11 [6]. The two optimized fillets reach an averaged life cycle which is eight times higher than for the circular fillet.

References


