Continuum structural design using local rule

T. Saito, T. Tamaki & E. Kita

School of Informatics & Sciences,
Nagoya University, Nagoya 464-8601, Japan.

Abstract

This paper describes the continuum structural design based on the concept of cellular automata. Cellular automata simulation is performed according to the local rule. The local rule is derived analytically from the objective functions and the constraint conditions. Derived rule is applied to the design of the continuum structures.

1 Introduction

Cellular automata are considered as the powerful tool for the simulation of the complex systems such as the liquid flow, the traffic flow and so on. This paper describes the application of the cellular automata simulation to structural design.

The cellular automata simulation is performed according to local rule. Definition of local rule is very important in the simulation and therefore, several schemes for definition of local rule have been presented. Authors presented the scheme to analytically derive the local rule from the optimization problem. In the scheme, the design domain is divided into small square cells and the cell thickness is taken as design variable. Penalty function is derived from the objective functions and the constraint conditions by introducing the weight functions and the penalty parameter. Stationalizing the function leads to the local rule to update design variable. In the scheme, the design domain must be uniformly divided into identical small square cells.
232  Computer Aided Optimum Design of Structures VIII

When the domain profile is very large or complicated, automatic mesh generator of commercial FEM software should be used for the discretization and in this case, it is very difficult to uniformly discretize the domain with identical cells. For overcoming this difficulty, this paper describes the extension of the local rule to non-uniformly discretized domain. Finally, the present scheme is applied to simple numerical example.

This paper is organized as follows. In the section 2, the related works are compared. In the section 3, finite element method is formulated briefly. In the section 4, the local rule is defined and then, the algorithm of the present scheme is described. In the section 5, the present scheme is applied to the shape determination of the continuum structure. Finally, the section 6 describes some conclusions.

2 Background

Basic Idea  Xie and his colleagues presented Evolutionary Structural Optimization (ESO) method[1, 2, 3]. In this scheme, the reference value is firstly specified. After the stress analysis by the finite element method, one removes the cells where the stresses are smaller than the reference value. Although the terminology “Cellular Automata” is not used in their method, it can be considered as one of the structural design method based on the concept of cellular automata.

The method employing the terminology “Cellular Automata” is firstly presented by Inou et. al.[4]. In the study, the design domain is divided into many small cells and equivalent stress distribution on the whole domain is estimated by the finite element method. Then, the reference stress which is individually specified at each cell is updated by applying local rule to the stress distribution. Since the local rule is derived from the numerical experience, the mathematical relationship between the rule and the optimization problem is not obvious.

Derivation of Local Rule  Cellular automata simulation is performed according to the local rule and therefore, it is very important to derive the local rule. Therefore, several derivation schemes are presented.

Oda et. al.[5] present Evolutionary Cellular Automata (ECA). The local rule is defined by using genetic algorithm or neural network. The rule is evolved by some numerical examples.

Payton et. al. employ as the local rule the following remodeling equation of human bone[6]:

\[
V_i \frac{\partial \rho_i}{\partial t} = \frac{\rho u K_i u}{2} - \frac{\xi_0}{\lambda} V_i
\]

where \( u, K_i \) and \( V_i \) denote the displacement vector, stiffness matrix and volume of the cell \( i \), respectively. \( \lambda, \gamma \) and \( \xi_0 \) mean Lagrange multiplier, the power law exponent and the parameter, respectively.
Kita et. al.[7] presented the mathematical derivation scheme of local rule. The objective functions and the constraint conditions are defined for optimization problem and then, the penalty function is derived. Stationalizing the penalty function leads to the local rule.

3 Finite element method

We shall explain briefly the finite element formulation for convenience of derivation of the local rule[8].

The principle of the virtual work without the body forces is given as

$$\int_\Omega \delta \varepsilon^T \sigma d\Omega = \int_{\Gamma_t} \delta u^T t d\Gamma$$  \hspace{1cm} (2)

where $\Omega$, $\Gamma_t$, and $\Gamma_i$ denote the domain occupied by the object under consideration, its displacement- and traction-specified boundaries, respectively. $u$, $t$ and $\sigma$ denote the vectors of the displacement, the traction, the strain and the stress components in the two-dimensional elastic problem, respectively. The physical quantities with a symbol $\delta$ mean the virtual ones and the superscript $T$ the transposition of the matrices and the vectors. The relationships between the physical quantities are given as

$$\varepsilon = Au$$ \hspace{1cm} (3)

$$\sigma = D\varepsilon$$ \hspace{1cm} (4)

$$t_i = \sigma_{ij}n_j$$ \hspace{1cm} (5)

where $n_j$ means the $x_j$-component of the outer normal vector on the boundary. $A$ and $D$ are defined as

$$A = \begin{bmatrix}
\partial / \partial x_1 & 0 \\
0 & \partial / \partial x_2 \\
\partial / \partial x_2 & \partial / \partial x_1
\end{bmatrix}$$ \hspace{1cm} (6)

$$D = \frac{E}{1 - \nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & (1 - \nu)/2
\end{bmatrix}$$ \hspace{1cm} (7)

where $E$ and $\nu$ denote Young’s modulus and Poisson’s ratio, respectively.

Discretizing the both sides of Eq.(2) by $N_e$ finite elements and $N_t$ boundary elements, we have

$$\sum_{e=1}^{N_e} \int_{\Omega_e} \delta \varepsilon_e^T \sigma_e d\Omega = \sum_{i=1}^{N_t} \int_{\Gamma_i} \delta u^T t d\Gamma$$ \hspace{1cm} (8)

The displacement components at the element $e$ are approximated by the interpolation functions $N$ with the nodal displacements $U_e$;

$$u_e = NU_e$$ \hspace{1cm} (9)
The strain and the stress components are expressed as

\[ \varepsilon_e = Au_e = ANU_e = BU_e \]  
\[ \sigma_e = De = DBU_e \]  

Substituting the above approximate expressions into Eq.(8), we have

\[ \sum_{e=1}^{N_e} \delta U_e^T h_e b_{e}^{T} U_e = \sum_{l=1}^{N_i} \delta U_i^T f'_l \]  

and then,

\[ Ku = f \]  

where \( h_e \) denotes the thickness and \( b_{e}^{T} \) the stiffness matrix at the element \( e \). \( f'_l \) and \( l \) denote the equivalent nodal force vector and the element number, respectively.

4 Local rule and optimization algorithm

Cell Relationship The local rule is defined as the local relationship between the neighboring cells (Fig.1). The cell-0 denotes the cell of which design variable is updated according to the local rule. All cells neighboring to the cell-0 are considered as the neighborhood cells to the cell. When some cells at the position of the neighboring cells do not exist, only remaining cells are considered as the neighborhood cells.

Objective Functions and Constraint Conditions We shall consider as the design objects the minimization of both the total weight of a structure and the deviation between the reference stress and the Von Mises equivalent stress at the cells. The local rule is derived by introducing the "CA-constraint condition", which is defined so as to minimize the variation of the equivalent stress of the neighboring cells with respect to the variation of the design variable.
Computer Aided Optimum Design of Structures VIII 235

The first objective function to minimizing the weight is defined as

\[ W_1 = \left( \frac{h_0}{h_0^0} \right)^2 = h_0^2 \]  

(14)

where \( h_0 \) and \( h_0^0 \) mean the thickness of the cell-0 and its initial value, respectively.

Assuming that the Von Mises equivalent stress and the yield stress of the material at the cell-0 are \( \bar{\sigma}_0 \) and \( \sigma_c \), respectively, the objective function for minimizing the stress deviation is defined as

\[ W_2 = \left( \frac{\bar{\sigma}_0}{\sigma_c} - 1 \right)^2 \equiv (\sigma_0 - 1)^2 \]

(15)

The CA-constraint conditions is defined as

\[ g_i = \frac{\bar{\sigma}_i}{\sigma_i^0} - 1 \equiv \sigma_i - 1 = 0 \quad (i = 1, \cdots, N_e) \]

(16)

where \( \bar{\sigma}_i \) and \( \sigma_i^0 \) denote the equivalent stresses at the neighboring cell \( i \) at the present and the former steps, respectively. \( N_e \) is total number of neighborhood cells to the cell-0.

**Functional of Optimization Problem** A new objective function \( W \) is defined from Eq.(14), (15) and (16) as follows:

\[
W = \alpha W_1 + \beta W_2 + p \sum_{i=1}^{N_e} g_i^2 \\
= \alpha h^2 + \beta(\sigma_0 - 1)^2 + p \sum_{i=1}^{N_e} (\sigma_i - 1)^2
\]

(17)

where \( p \) means the penalty function and \( \alpha \) and \( \beta \) the weight parameters defined as

\[
\alpha + \beta = 1 \quad , \quad \beta = \begin{cases} \sigma_0^0 & (\sigma_0 < 1) \\ 1 & (\sigma_0 \geq 1) \end{cases}
\]

(18)

Taylor’s expansion representation of Eq.(17) leads to ;

\[
W(h_0 + \delta h_0) \approx \alpha(h_0 + \delta h_0)^2 + \beta(\sigma_0 + \bar{\sigma}_0 \delta h_0 - 1)^2 \\
+ p \sum_{i=1}^{N_e} (\sigma_i + \bar{\sigma}_i \delta h_0 - 1)^2
\]

(19)

where \( (\cdot) = \partial/\partial h \). By vanishing the first variation of Eq.(19), we have

\[
\delta h_0 = - \frac{\alpha h_0 + \beta(\sigma_0 - 1)\bar{\sigma}_0 + p \sum_{i=1}^{N_e} (\sigma_i - 1)\bar{\sigma}_i}{\alpha + \beta \bar{\sigma}_0^2 + p \sum_{i=1}^{N_e} \bar{\sigma}_i^2}
\]

(20)
Approximate Design Sensitivity  
Equation (20) includes the design sensitivity \( \dot{\sigma}_i \). So, we shall consider here approximate estimation of the design sensitivity. If the principle of the virtual work is assumed to be valid at each cell, we have

\[
\int_{\Omega_e} \delta \varepsilon^T \sigma d\Omega = \int_{\Gamma_e} \delta u^T \dot{t} d\Gamma
\]  

(21)

Approximating the physical quantities by the interpolation functions with nodal values, we have

\[
hK'_{e} U = f
\]

Since the specified values on the boundary are independent of \( h_0 \), direct differentiation of the above equation with respect to \( h \) leads to

\[
K'_{e} U_e + h K'_{e} \dot{U}_e = f = 0
\]

\[
\dot{U}_e = -\frac{1}{h} (K'_{e})^{-1} K'_{e} U_e \equiv -\frac{1}{h} U_e
\]  

(22)

On the other hands, direct differentiation of both sides of Eq.(11) with respect to \( h \) leads to the sensitivities of the stress components with respect to \( h \);

\[
\dot{\sigma}_e = DB \dot{U}_e = -\frac{1}{h} DBU_e = -\frac{1}{h} \sigma_e
\]  

(23)

Finally, the stress sensitivity \( \dot{\sigma}_i \) in Eq.(20) can be approximated as

\[
\dot{\sigma}_i = -\frac{\sigma_i}{h}
\]  

(24)

Substituting Eq.(24) into Eq.(20), we have

\[
\delta h_0 = \frac{[\alpha h_0^2 + \beta (\sigma_0 - 1)\sigma_0 + p \sum_{i=1}^{8} (\sigma_i - 1)\sigma_i]h_0}{\alpha h_0^2 + \beta \sigma_0^2 + p \sum_{i=1}^{8} \sigma_i^2}.
\]  

(25)

Local Rules and Algorithm of the Scheme  
The thickness of the cell-0 \( h_0 \) is updated by adding \( \delta h_0 \) as follows:

\[
h_0^{k+1} = h_0^k + \delta h_0
\]  

(26)

The algorithm of the present scheme is as follows.

1. Input the initial data; the domain size, the number of the cells and the boundary and the design conditions.
2. Perform the stress analysis by the finite element method.
3. Check the convergence criterion. If the criterion is satisfied, the process is terminated. If not so, the process goes to the next step.
4. The cell thickness is updated according to the rule.
5. Go to Step 2.
Fig. 2: Numerical example 1

(a) No. of cells = 16 × 24
(b) No. of cells = 24 × 36
(c) No. of cells = 32 × 48
(d) No. of cells = 40 × 60
(e) No. of cells = 48 × 72

Fig. 3: Final profiles obtained different initial meshes
5 Numerical example

Example 1 A first example is shown in Fig. 2. The object domain is uniformly divided into cells of different size. Physical parameters are specified as Young’s modulus $E = 10^{10} (Pa)$, Poisson ratio $\nu = 0.2$, initial cell thickness $h_0 = 10^{-2} (m)$, load $P = 10^3 (N)$, the penalty parameter $p = 1$ and the permissible stress $\sigma_c = 80\%$ of initial maximum stress. The final profiles obtained after 10000 iteration steps are compared in Fig. 3. Numbers of cells at the cases are a) $16 \times 24$ (16 cells in vertical direction and 24 ones in horizontal direction), b) $24 \times 36$, c) $32 \times 48$, d) $40 \times 60$ and e) $48 \times 72$, respectively.

We notice that the final profiles have the same topology although the numbers of cells are different from each other.

Example 2 A second example is a plate with a circular hole as shown in Fig. 4. Edge of the hole is fixed in all directions and the central load is applied to the right. Automatic mesh generator of commercial FEM software generates the finite element mesh as shown in Fig. 5. We notice that the mesh generated by the automatic software is non-uniform. The present
scheme is applied to this problem. Physical parameters are specified as Young's modulus \( E = 10^{10}(Pa) \), Poisson ratio \( \nu = 0.2 \), initial cell thickness \( h_0 = 10^{-2}(m) \), load \( P = 10^3(N) \), the penalty parameter \( p = 1 \) and the permissible stress \( \sigma_c = 80\% \) of initial maximum stress.

Profiles of the object under consideration at 1st, 100th, 500th and 2000th iteration steps are shown in Fig. 6.

### 6 Conclusions

This paper describes the structural design scheme using local rule employed in the cellular automata simulation. In the previous study, the object domain is uniformly divided into square cells and the local rule is defined on the relationship among uniform cells. However, if the commercial software is employed for mesh generation, the mesh is usually non-uniform. Therefore, in this paper, the local rule is extended for that for non-uniform mesh.

The present scheme is applied to numerical examples. In the first example, the optimization process started from uniform mesh divided with different numbers of cells and final shapes with identical profile could be obtained. In the second example, non-uniform mesh is considered as initial one. The results were satisfactory. We can say the extended rule is adequate for these problems.
References


