Optimization of shear wall allocation in 3D frames by branch-and-bound method

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Abstract

It is required that distribution of shear resistant elements such as walls or braces is appropriately accomplished in order to decrease torsional moment about a vertical axis due to lateral forces such as seismic or wind loads, which lead to local large displacements. An allocation design of shear walls in a multi-storied 3D building system, which is reduced to a design problem of appropriate selections of walls from a large number of discrete candidates, is a combinatorial optimality problem. Such a structural design is subjected to not only structural but also architectural and constructional constraints. In addition, when the scale of the design problem becomes larger, an effective scenario of computer support is indispensable to obtain rational design solutions. The present study deals with a combinatorial searching problem stemmed from the optimal allocation of shear walls in 3D multi-story frames subject to torsional moment about a vertical axis by application of the branch-and-bound method. The present problem is formulated as two problems; the searching problem for feasible wall allocations at each story and the minimization of the total amount of wall along the story. It is illustrated with some practical design examples that optimal solutions can efficiently be obtained by the present optimization method.

1 Introduction

The economical growth after the devastation by World War II in Japan has realized construction of a large number of both low and medium rise buildings in the urban area with the result of structural characteristics including geometrically irregular figures in floor and elevation which spontaneously provide torsional moment about a vertical axis when subjected to lateral
forces due to strong earthquakes and its amplification on the thick alluvium. Structural design of such buildings must include appropriate allocation of shear resistance elements against such a torsional moment at the preliminary design stage. Furthermore, recent damage investigation of such buildings due to strong earthquakes suggests requirement of further reinforcement with additional allocation of shear resistance elements in the corresponding 3D building frame. In Japan such building structures consist frequently of a 3D bare frame, which is a rigid jointed frame, with larger drift and an appropriately allocated shear resistance elements with smaller drift. Hence, one of the important structural design problems is search for an optimal allocation of such shear elements under various constraints to minimize torsional moment about a vertical axis due to lateral loading. The allocation of shear elements can be treated as the topology optimization problem, whose methods are firstly applied to the layout design of bracing systems to multi-story steel frames by Mijar [1] and Liang [2]. The present allocation problem is a decision problem of appropriate allocation of shear elements between a large numbers of discrete allocation candidates, and becomes a typical combinatorial optimality problem. Hager and Balling [3] minimize total weight of a planar frame by the branch-and-bound method based on the linear programming. The enhancing branch-and-bound method with the sequential quadratic programming (SQP) is realized to the minimum weight design of truss structures by Tseng [4], where the neighboring searching techniques are applied. This can search for discrete solutions near the continuous optimum solution obtained by the linear or non-linear programming. Bremicker [5] applies the branch-and-bound method with SQP to the minimum weight design of truss structures with discussion focused on the relationship between accuracy of solutions and CPU time. The genetic algorithm (GA) method is applied to such discrete structural optimization [6, 7]. The present study deals with numerical comparison of effectiveness of solutions and CPU time regarding the optimal allocation of shear walls in a 3D rigid frame subjected to such a torsional moment by means of the branch-and-bound method.

2 Formulation of optimal allocation of shear walls

The present problem can be described as a search problem for shear wall allocations, which minimizes the total weight of walls. Three types of allocations such as empty, removable, and coerced shear walls are assumed as shown in Figure 1. Assuming that $M$-storied frame has a uniform story height, the present problem can be formulated as follows.

\[
A_w = \sum_{i=1}^{M} A(L_i) \rightarrow \text{min.} \\
\text{subject to } g_k(L_1, \ldots, L_M) \in C_k \ ; \ k = 1, 2, \ldots
\] (1)
Figure 1: An example plan and its allocation condition

where, $L_i$ means a wall allocation at the $i$-th story, which is a design variable vector, $A(L)$, the sum of section area of a corresponding allocation, $g_k(\cdot)$, the $k$-th constraint function, and $C_k$, a feasible region of the $k$-th constraint, respectively. For $N$ candidate allocations in each story, the design variable vector can be expressed as follows.

$$L_i = \{L_{i1}, \ldots, L_{iN}\}, \quad L_{ij} \in \{1, 0\}; \quad j = 1, \ldots, N \tag{2}$$

where, $L_{ij}$ represents a binary variable, which takes 1 when $j$-th wall is allocated, and 0, not allocated, respectively. The practical constraints are discussed in the subsequent section.

Optimization for sizable structures becomes difficult because of enormous searching space. Herein, two steps are assumed that after partial search for feasible solutions at each story, overall minimization of the total weight of walls along the story is accomplished. Consequently, eqn (1) can be divided into the following two problems.

$$P_0 : A_w = \sum_{i=1}^{M} A(L_i) \rightarrow \text{min.} \quad \left\{ \begin{array}{l}
\text{subject to } L_i \in \{L_i^*\}; \quad i = 1, \ldots, M \\
\end{array} \right. \tag{3}$$

$$iQ_0 : \text{find } L_i^* \quad \left\{ \begin{array}{l}
\text{subject to } g_k(L_i^*) \in C_{k,i}; \quad k = 1, 2, \ldots \\
\end{array} \right. \tag{4}$$

where, $C_{k,i}$ is a feasible region of the $k$-th constraint at the $i$-th story, which depends on the lower story allocations. $iQ_0$, whose design variables are $L_{ij}(j = 1, \ldots, N)$, is a satisficing problem under the $i$-th story constraints. For the optimization problem, $P_0$, the wall allocation at each story should be selected among a set of feasible allocations, $\{L_i^*\}$, which is obtained from $iQ_0$. Thus, both $iQ_0$ and $P_0$ are typical combinatorial problems.

3 Practical constraints

3.1 Strength constraints

The present frame must be subjected to the constraints from structural strength. Thus, in the $x$ and $y$ directions, the ultimate strength at each
story should satisfy the following constraint.

\[ iQ_u \geq iQ_{un} \]  \hspace{0.5cm} (5)

where \( iQ_u \) is the \( i \)-th story strength, and \( iQ_{un} \), the corresponding design shear force, respectively, which are practically defined as follows.

\[ iQ_u = iQ_{uw} + 0.7iQ_{uc} \]  \hspace{0.5cm} (6)

\[ iQ_{un} = D_sC_iW_i \]  \hspace{0.5cm} (7)

where \( iQ_{uw} \) means the sum of strength of brittle members including shear walls, \( iQ_{uc} \), the sum of strength of ductile members, \( D_s \), the structural factor to reduce design shear forces taking into account ductility, \( C_i \), the story shear coefficient, and \( W_i \), the total weight of building from the \( i \)-th story to the top, respectively. Eqn (6) means that \( iQ_u \) is defined as a shear resistance when brittle members reach their ultimate strength, and then ductile members are assumed to reach 70% of their ultimate strength.

### 3.2 Displacement constraints

To ensure both story drift tolerance and uniform story drift ratio in the \( x \) and \( y \) directions independently, the following constraints should be satisfied.

\[ \delta_i = C_iW_i/K_i \leq \delta_a \]  \hspace{0.5cm} (8)

\[ (1 - \varepsilon)\delta_M \leq \delta_i \leq (1 + \varepsilon)\delta_M \]  \hspace{0.5cm} (9)

where \( K_i \) denotes the \( i \)-th story lateral stiffness, \( \delta_a \), the allowable drift, \( \varepsilon \), the allowable deviation, and \( \delta_M \), a mean drift of the whole story, respectively.

### 3.3 Torsional constraints

To decrease torsional responses, the following constraints are established at each story.

\[ \hat{\alpha}_d e_x/r_{ey} \leq R_{da}, \text{ and } \hat{\alpha}_d e_y/r_{ex} \leq R_{da} \]  \hspace{0.5cm} (10)

where \( e_x \) and \( e_y \) mean eccentricities between the centers of gravity and stiffness in the \( x \) and \( y \) directions, \( r_{ex} \) and \( r_{ey} \), radiiuses of torsional stiffness, and \( R_{da} \), the corresponding torsional tolerance, respectively. \( \hat{\alpha}_d \) is a dynamic amplification of eccentricity, which is defined as follows.

\[ \hat{\alpha}_d = \frac{2}{1 - R^2 + \sqrt{(1 - R^2)^2 + 4(\overline{\varepsilon}_x^2 + \overline{\varepsilon}_y^2)/\overline{\tau}_c^2}} \]  \hspace{0.5cm} (11)

\[ R^2 = (1 + \overline{\varepsilon}_x^2 + \overline{\varepsilon}_y^2)/\overline{\tau}_c, \text{ and } \overline{\tau}_c = \max\{\overline{r}_{ex}, \overline{r}_{ey}\} \]  \hspace{0.5cm} (12)

where tildes, \( \sim \), means a ratio with a mass radius of gyration at each story.
3.4 Up-lift constraints

A large overturning moment due to lateral loads provides the up-lift effect at foundations which must be prevented. Thus,

\[(1 + \alpha_F) \frac{A_{C_j}}{A_{F}} W_1 \geq N_{0j}; \ j = 1, ..., N\]

where \(\alpha_F\) represents the weight ratio of foundation, \(A_{C_j}\), a supporting floor area of the \(j\)-th column, \(A_{F}\), the total floor area at each story, and \(N_{0j}\), a tension of a pile due to lateral loads, respectively.

3.5 Allocation constraints

Shear walls should be located continuously along the story from the structural point of view. The most important constraint is prevention from violation of floor layout by shear walls regarding the cores of fire proofing zone, of elevator pit, of stair case and noise insulation zone. This requires three flags of allocation categorized as empty, removable and coerced shear walls.

4 Application of branch-and-bound method

4.1 Branch-and-bound method

The branch-and-bound method is a discrete optimization method, which searches for optimum solutions by repetitive implementation of the branching and bounding operations. The branching operation is decomposition of a discrete problem into some partial problems, which have less design variables than the mother problem. For example, the following discrete optimality problem, \(P_0\),

\[
P_0 : f(x_1, ..., x_n) \rightarrow \min, \\
\text{subject to } g_i(x_1, ..., x_n) \geq b_i; \ i = 1, ..., m \\
x_j \in \{1, 0\}; \ j = 1, ..., n
\]

(14)

can be decomposed into the following two partial problems, \(P_1\) and \(P_2\), by setting \(x_1 = 1\) or \(x_1 = 0\).

\[
P_1(x_1 = 1) : f(1, x_2, ..., x_n) \rightarrow \min, \\
\text{subject to } g_i(1, x_2, ..., x_n) \leq b_i; \ i = 1, ..., m \\
x_j \in \{1, 0\}; \ j = 2, ..., n
\]

(15)

\[
P_2(x_1 = 0) : f(0, x_2, ..., x_n) \rightarrow \min, \\
\text{subject to } g_i(0, x_2, ..., x_n) \leq b_i; \ i = 1, ..., m \\
x_j \in \{1, 0\}; \ j = 2, ..., n
\]

(16)
Obviously, the following relationship between minimum objective function values of these problems, which are described by $F(P_i)$, is established.

$$F(P_0) = \min\{F(P_1), F(P_2)\}$$  \hspace{1cm} (17)

The branching operation can also be applied to both $P_1$ and $P_2$. As a result, $2^n$ partial problems are obtained due to recursive branching operations. Because the search only by the branching operation is equivalent to the exhaustive enumeration method, pruning futile alternatives by the following criteria is required for the present method.

- When the optimal solution is obtained from a partial problem, $P_i$, it is not necessary to extend further branching operation from $P_i$.
- If a partial problem cannot provide the optimal solution of original problem, it is not necessary to extend further branching operation.

These criteria to halt further branching is called as the bounding operation, which thus can terminate the partial problem, $P_i$. In general, practical implementation of the bounding operation can be accomplished by either the lower bound test or the dominance test.

### 4.2 Minimization of the total weight of walls along the story

When the wall allocation of the first story is assumed $L_1^{(k)} \in \{L_i^*\}$, the following partial problems, $P_k (k = 1, 2, \ldots)$, can be obtained from the original problem, $P_0$.

$$P_k : A_w^{(k)} = A(L_1^{(k)}) + \sum_{i=2}^{M} A(L_i) \longrightarrow \min.$$  \hspace{1cm} (18)

subject to $L_i \in \{L_i^*\}; \ i = 2, \ldots, M$

The minimum amount of walls, $A_{opt}$, can be obtained as follows.

$$A_{opt} = \min_k A_{opt}^{(k)}$$  \hspace{1cm} (19)

where $A_{opt}^{(k)}$ means the minimum amount of walls assuming that the first story allocation is $L_1^{(k)}$. Successively, the following partial problems, $P_l$, whose wall allocations at lower than the $(m - 1)$-th story are already determined, can be obtained.

$$P_l : A_w^{(l)} = \sum_{i=1}^{m-1} A(L_i^{(l)}) + \sum_{i=m}^{M} A(L_i) \longrightarrow \min.$$  \hspace{1cm} (20)

subject to $L_i \in \{L_i^*\}; \ i = m, \ldots, M$

Thus the bounding operations are implemented, if either one of the following conditions.
1. When it is proved that no feasible allocation exists at any story higher than the \( m \)-th story, the partial problem, \( P_i \), should be terminated. This is called as the existence test of feasible solutions.

2. When the following relationship is satisfied, \( P_i \) should be terminated, which is called as the lower bound test.

\[
\sum_{i=1}^{m-1} A(L_i^{(i)}) + A_{nec,m} > A^*	ag{21}
\]

where \( A_{nec,m} \) represents a necessary amount of shear walls at higher than the \( m \)-th story, which can be determined by the strength constraints, and \( A^* \), an incumbent value or the minimum weight during searching process, respectively.

### 4.3 Search for feasible wall allocations at each story

The search problem for feasible allocations, \( iQ_0 \), can be decomposed into two partial problems by setting \( L_{ij} = 1 \) or \( L_{ij} = 0 \). Since this problem is a satisficing problem, the lower bound test cannot be applied. Herein, only the existence test of feasible solutions under the strength and displacement constraints is implemented. For example, partial problems can be terminated when the following equation of the ultimate strength is satisfied.

\[
\sum_{j \in J^+} iQ_{uw,j} + \sum_{j \in J^+} iQ_{uw,j} + 0.7iQ_{uc} < iQ_{un}	ag{22}
\]

where \( iQ_{uw,j} \) means the ultimate strength of the \( j \)-th shear wall, \( J^+ \), a set of suffix, which is defined as \( J^+ = \{j | L_{ij} = 1, j = 1, \ldots, n\} \), and \( J \), a set of suffix of free variables in the corresponding partial problem, respectively.

### 5 Design examples and discussion

#### 5.1 Influence of floor area on searching efficiency

Optimal allocations by the present method is applied to 3 types of 4-storied frames with different floor area by Figure 2 for the purpose of comparison of CPU times. Two types of allocations such as empty in broken lines and removable walls in solid lines are prescribed. The number of total design variables of each frame are 32, 64 and 96, and then, the numbers of combinations become \( 2^{32} \), \( 2^{64} \) and \( 2^{96} \), respectively. Moreover, it is assumed that each story height is 3.5(m), the structural factor, \( D_s = 0.55 \), the allowable story drift, \( \delta_s = 1/400 \), the allowable deviation of drift, \( \varepsilon = 0.5 \), and the torsional tolerance, \( R_{da} = 0.2 \), respectively, from the practically structural point of view in Japan.

All optimal allocations, one of which is illustrated in Figure 3, are obtained for each example frame. Figure 4 shows the relationship between the number of design variables and CPU time ratio, which is defined as a ratio of CPU
5.2 Influence of the number of stories on searching efficiency

Three types of frames, whose numbers of stories are 4, 5 and 6, respectively, are optimally designed. Both the shape and the allocation constraint of each floor are equal to the previous example of $3 \times 3$ bayed frame. The size time to that consumed by the $2 \times 2$ bayed frame example. Although the CPU time exponentially increases to the number of design variables, increment of time is prominently less than that of the exhaustive enumeration method.
Table 1: the size of members of 5 and 6-storied frames

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<th>$B_c \times D_c$</th>
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of members of 5 and 6-storied frames are shown in Table 1. The numbers of design variables of each frame are 64, 80 and 96, respectively. Figure 5 shows the relationship between the number of design variables and CPU time ratio, which is defined as a ratio CPU time to that of the 4-storied frame example. It can be also seen that the increment of CPU time is considerably less than the case of the exhaustive enumeration method.

Figure 6 provides comparison of increments in CPU time between the case in which the size of floor increases from $3 \times 3$ to $4 \times 4$ for the 4-storied frame and the case in which the number of stories increases from 4 to 6 for the $3 \times 3$ bayed frame. The number of design variables increases from 64 to 96 in both cases. It is understood that increase in stories is required more CPU time than that of increase of floor area. This implies that many shear walls are required at upper stories in order to satisfy the stiffness constraints, subsequently, the effect of the lower bound test decreases.

![Figure 5: Relationship between the number of variables and CPU time](image1)

![Figure 6: Comparison of increments in CPU time](image2)
6 Concluding remarks

The optimal allocation of shear walls in 3D frames against large torsional moment can be, herein, described by two problems: search for feasible solutions at each story and minimization of the total wall weight. This can be accomplished by the present branch-and-bound method, which can find rigorous solutions differently from other approximate methods such as the GAs. The following remarks are obtainable.

- Applying the lower bound test or the existance test discussed in the present study, the rigorous optimal allocations can be obtained with reasonable CPU time in comparison with the exhaustive enumeration. This implies that even for discrete optimization problem, rigorous optimal solutions can be obtained efficiently by means of appropriate bounding operations.

- Because of difference in the effect of bounding operations, the necessary computing times are different even for problems with the same numbers of design variables. From the present numerical design examples, it is recognized that increase in stories is required more CPU time than that of increase of floor area.

References


