STRUCTURAL RELIABILITY ANALYSIS BY UNCONSTRAINED OPTIMIZATION AND MULTIMODAL MONTE CARLO CONDITIONAL IMPORTANCE SAMPLING

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Abstract — This paper is concerned with a two steps computational approach for structural system reliability analysis: First, the search for points of maximum likelihood is reduced to an unconstrained optimization problem, using a conditional expectation technique coupled with an adaptive response surface interpolation function, defined over the integration domain and successively updated to take into account the previously calculated finite regions associated with local maxima. Second the failure probability is obtained by an effective utilization of the multimodal Monte Carlo importance sampling variance reduction technique over a finite number of discrete regions on the integration domain. Some results illustrate the main features of the proposed approach.

INTRODUCTION

For time-invariant problems, the probability of failure, \( P_f \), is given by the n-fold integral

\[
P_f = \int_{G(X) \leq 0} f_X(x) \, dx
\]

(1)

The vector of basic random variables \( X = (X_1, X_2, \ldots, X_n)^T \) represents the uncertain quantities such as loads, material properties, structural member...
dimensions, etc., and $f_X(x)$ is the joint probability density function of $X$. A limit-state function $G(X)$ is used to define the limit-states for which the structure no longer satisfies their requirements. By convention, the limit-state function is formulated such that $G(x) \leq 0$ when the structure fails by any limit-state violation and $G(x) > 0$ when the structure is safe. The limit-state surface $G(x) = 0$ is the separation surface of the failure region from the safe region in the vector space spanned by $X$.

Real structures are usually submitted to several limit-states, such as ultimate loads, fatigue damage, serviceability, etc. Therefore the system limit-state function $G(x)$ could be expressed, for instance, as the union of all limit-states considered in the analysis, and the corresponding limit-state surface, $G(x) = 0$, will be the envelope surface of the failure region $G(x) \leq 0$.

The numerical integration of Eq. (1) has led to many Monte Carlo algorithms available in literature such as importance sampling in Cartesian coordinates and directional simulation in polar coordinates.

For importance sampling algorithms, the sampling should be concentrated in regions associated to maximum likelihood points, satisfying the limit-state surface $G(x) = 0$. Suggestions to make up importance sampling functions, as Gaussian distributed were proposed by Melchers or as Gaussian correlated to match the basic random variables by Schueller and Stix. The search for maximum likelihood points in conjunction with the $P_f$ calculation, using adaptive sampling, were suggested by Schueller et al. and through interpolation techniques by Melchers. In order to reduce the number of limit state function evaluations, the use of simplified response surfaces have been advocated by Schueller et al. and grouped sampling or censored sampling by Melchers.

For directional simulation, the vector $X$ of basic random variables, in the $n$-dimensional real space $R^n$, is expressed through a polar coordinates system centered at the mean point $\mu_X$ of the basic random variables. The vector $X$ is then transformed by $X = RA$, where $R$ is a random variable defined such that $R = 0$ at $\mu_X$, and $A$ is a unit vector that is distributed in order to be defined in a unit sphere in $R^n$. For standard Gaussian joint probability distribution $f_X(x)$, $R^2$ will be chi-square distributed. This approach was proposed by e.g. Bjerager and Ditlevsen. The extension to non-Gaussian $f_X(x)$ was also made by Ditlevsen. A further generalization, to choose appropriate sampling distribution in radial direction and concentrating sampling in regions of most interest, was proposed by Melchers.

In Cartesian coordinates importance sampling has been very effective in Monte Carlo integration techniques for structural reliability analysis.
However it has been stated that the method may be not well-suited for problems having one or more limit-state functions such has to coincide (at least in part) with a contour of the joint probability density function \( f_x(x) \). This situation, typical for circular, cylindrical, etc. limit-state functions, may give rise to a large number (even infinity) of local maximum likelihood points. Noting that the number of limit state function evaluations should be kept as low as possible, due to the complexity of the numerical calculations involved in realistic problems. The importance sampling in rectangular coordinates seems then to be relegated to problems for which the points of local maximum could be identified and/or are not to numerous.

To overcome these drawbacks, extensions to polar coordinates have been developed, resulting the previously mentioned directional simulation methods. In spite of the natural elegance of this approach the authors believe that the importance sampling in Cartesian coordinates can be also quite attractive, if it is properly used in conjunction with an adaptive response surface interpolation function, that enable the application of an unconstrained maximization routine to identify a finite number of discrete regions over the integration domain to be covered by a multimodal importance sampling function. In this way even the circular, cylindrical, etc. limit-state functions, that give rise to an infinity number of maximum likelihood points could be treated with a reasonable small number of limit state function evaluations, as will be described herein.

**CONDITIONING**

To apply the conditional expectation approach in Eq. (1) the conditioning variables should be statistically independent. This limitation could be removed by a suitable transformation of a dependent problem into one with independent random variables.

Let the \( n \)-dimensional \( X \) of dependent random variables be expressed as:

\[
X = T U
\]  

(2)

where \( U \) is the standard Normal (Gaussian) vector of statistically independent random variables and \( T \) is the well known transformation matrix.

The probability of failure can than be written in the \( U \) space, without any loss of generalization, as:

\[
P_f = \int_{G(U) \leq 0} \prod_{i=1}^{n} \phi(U_i) \, dU_i
\]  

(3)
where \( \phi \) is the probability density of the standard Gaussian distribution.

Denoting by \( U_n \) a variable with non zero sensitivity, the order of integration in Eq. (3) can be reduced by one from \( \mathbb{R}^n \) to \( \mathbb{R}^{n-1} \) vector space, resulting:

\[
P_f = \int_{\mathbb{R}^{n-1}} \Phi[U_n(U_i)] \ast \prod_{i=1}^{n-1} \phi(U_i) \, dU_i
\]

where \( \Phi \) is the cumulative standard Gaussian distribution.

It should be noted that the integration domain of Eq. (4) is now extended over the entire \( \mathbb{R}^{n-1} \) vector space. Therefore the constrained maximization problem, associated with the Eq.(1), was transformed into an unconstrained, one making possible the application of more straightforward random search optimization algorithms. The \( U_n \) variable in Eq. (4) is expressed as a function from the remaining \( n-1 \) variables \( U_i \) as stated in the next section.

**ADAPTIVE RESPONSE SURFACE INTERPOLATION FUNCTION**

The variable \( U_n \) in Eq. (4) is calculated from the \( n-1 \) variables \( U_i \) through the following relationship:

\[
U_n = T^{-1}[X_n( X_i [ T U_i ] )]
\]

where \( T \) is the transformation matrix in Eq. (2) and \( T^{-1} \) is the corresponding inverse transformation. The values of \( X_n \) are calculated from \( X_i \) as a limit-state surface approximation function to the envelope surface \( G(X) = 0 \) of the failure domain. This equation is solved to extract the values of \( X_n \), for an initial pre-set number of points to span a sufficient large region in \( \mathbb{R}^{n-1} \), in order to encompass all possible points of local maximum likelihood. Therefore a \( n \)-dimensional limit-state surface approximation function is constructed and expressed as:

\[
X_n = F(X_{n-1})
\]

In multidimensional interpolation it is normally seek an estimate of \( X_n \) in Eq. (6) from an \( n-1 \) dimensional grid of \( m \) tabulated values of \( X_n \). It is easy to realize that the number of grid points (structural analysis), equal to \( m^{n-1} \), will be of no practical interest for \( n >> 4 \). Therefore an adaptive approach was used instead, starting with \( 1+(m-1)(n-1) \) interpolation points located in the coordinates axis. This initial grid is then successively updated to encompass the regions associated to the maximum likelihood points.
Let's consider for instance the 2-dimensional grid, generated by \( m = 5 \) interpolation points for each dimension, as it is shown in Figure 1. Normally it would be necessary \( m^2 = 25 \) grid points, but the interpolation function is started with only \( 1 + (m-1)2 = 9 \) points located on the coordinate axis:

\[
Z_3(\mathbf{Z}) = Z_3(0) + \varphi_1(Z_1) + \varphi_2(Z_2)
\]  

(7)

where \( Z_i \) is the \( X_i \) random variable normalized to zero mean and unity standard deviation, \( Z_3(0) \) is the \( Z_3 \) value for the point \((Z_1=0, Z_2=0)\) and \( \varphi_i(Z_i) \) is a Cubic Spline interpolation function defined on the \( Z_i \) axis.

![Figure 1 - 5 x 5 grid](image1.png)

Figure 1 - 5 x 5 grid

The interpolation function is then successively updated to take into account the regions associated with local maxima through the addition of corrective error functions.

![Figure 2 - Maximum Point](image2.png)

Figure 2 - Maximum Point
Let \( P(p_1,p_2) \) be the coordinates of a maximum point as shown in Figure 2.

If \( \Delta_p \) is the difference error between the interpolated and the calculated values, the update of the Eq. (7) is made by the addition of the following correction term:

\[
\Delta_p \sqrt{\prod_{i=1}^{2} \zeta_i(Z_i)}
\]  

where \( \zeta_i(z_i) \) are Cubic Spline interpolation functions with unity value for the maximum point \( P \) and zero values in the interval range points \((a_i, b_i)\) both for the function and for its first derivative, as indicated in Figure 3:

![Figure 3](image-url)

The generalization to \( n \)-dimensions, \( m \) interpolations points and \( r \) regions associated with local maximum points results the expression:

\[
Z_n(Z) = Z_n(0) + \sum_{i=1}^{n} \varphi_i(Z_i) + \sum_{k=1}^{r} \Delta_k \sqrt{\prod_{i=1}^{n} \zeta_i(z_i)}
\]  

It worth wile to mention that it is always possible to work with a finite number of regions associated with local maximum points (even in infinite number), due to the use of the multimodal Monte Carlo importance sampling integration. The minimum size of each region is defined by the radius of a hyper-sphere in the \( n \)-dimensional U-Space. The value of this radius is dictated by the precision required in the integration. Limited experience suggest a value in the range of two standard deviations could be adequate for multimodal Gaussian importance sampling functions in U-Space.

**UNCONSTRAINED OPTIMIZATION ALGORITHM**

Among the several unconstrained optimization algorithms presented in the literature, a simple random search unconstrained multidimensional maximization method was used in this paper. It's worth mention that the search for the regions associated with local maximum points can be made in
the U-Space with the unconstrained objective function defined in Eq. (4). The local maximum point search was then performed in a successive way for the points situated outside the union of the hyper-spheres of radius equal to two standard deviations and centered in the previous founded local maximum points. Adopting this strategy it is possible to overcome the drawbacks of problems with infinite number of local maximum points.

MULTIMODAL MONTE CARLO IMPORTANCE SAMPLING INTEGRATION

The integration of the Eq. (4) can be made by the well known multimodal Monte Carlo importance sampling integration, using multimodal independent Gaussian importance sampling functions:

\[ P_f = \frac{1}{N_p} \sum_{j=1}^{N_p} \Phi[v_n(v_j)] \ast \prod_{i=1}^{n-1} \phi(v_{ij}) \]

\[ \frac{\sum_{k=1}^{r} \omega_k \prod_{i=1}^{n-1} \phi(v_{ij} - p_{ik})}{\sum_{k=1}^{r} \omega_k \prod_{i=1}^{n-1} \phi(v_{ij} - p_{ik})} \]

where \( N_p \) is the number of sample points, \( n \) is the number of dimensions, \( r \) is the number of regions, associated with the local maximum points, \( \omega_k \) is the weight of each importance sampling function, selected as independent Gaussian distributed and shifted to the maximum probability density point \( k \) with \( p_i \), coordinates and \( v_{ij} \) is the \( j \) vector of sample values \( v_i \) generated by the multimodal importance sampling function.

NUMERICAL APPLICATION

Consider a simple example of a cylindrical limit state function \( G(x,y,z) = 9 - x^2 - y^2 - z \) in standard Gaussian random variables for which known results are available.

The objective function, defined in Eq. (4) is shown in Figure 4 and the corresponding multimodal importance sampling for a radius equal to three standard deviations is represented in Figure 5.

The coordinates of the calculated maximum local points, associated with 4 spherical regions of radius equal to 3, are given in Table 1. It was used a response surface interpolation function, starting with 9 preset points. This initial interpolation function was successively updated with 4 additional points to encompass the region associated to the maximum likelihood local points, resulting a total of 13 limit functions evaluations.
The value of failure probability, obtained with 10,000 samples, was $P_f = 0.014$ and the corresponding value, calculated by numerical integration, was $P_f = 0.0126$ for a tolerance of $10^{-5}$.

Figure 4 - Objective Function for a Cylindrical Limit State

Figure 5 - Multimodal Importance Sampling
Coordinates | Point 1 | Point 2 | Point 3 | Point 4
---|---|---|---|---
X | 3.09 | 0.00 | -0.04 | -3.09
Y | 0.00 | 3.10 | -3.08 | 0.08

Table 1 - Coordinates of the Maximum Local Points

CONCLUSIONS

A structural reliability analysis, using a very simple approach to solve the unconstrained optimization problem, was proposed. The use of an adaptive response surface interpolation function and the problem formulation as an unconstrained optimization one set up an effective and very efficient application of multimodal Monte Carlo importance sampling technique, even in the case of an infinite number of local maximum points. Finally, it is worth mentioning that a better precision in the failure probability evaluation can be obtained by reducing the radius of the hyper-spherical regions associated to the local maximum likelihood points.

REFERENCES