UNCERTAINTY REPRESENTATION MODELS IN THE
FINITE ELEMENT ANALYSIS

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Abstract - Uncertainty in structural engineering analysis exists in the
architecture of a structural system, its basic parameters, the information
resulting from the abstracted aspects of the system, and the non abstracted or
unknown aspects of the system. Also, uncertainty is present as a result of
prediction models, analysis and design of structures, and general lack of
knowledge in the behavior of real structures.

INTRODUCTION

The uncertainty of a system parameter may be attributed to cognitive and non-
cognitive sources. The cognitive sources include uncertainty due to imprecise
and vague information and are submitted to subjective judgment, the
probability and statistics axioms are limited to treat this type of uncertainty, so
the theory of Fuzzy Sets may be applied. The non-cognitive sources include
the system parameters that are treated as random variables with known
probability distributions, in these cases the response of the system can be
determined using theory of probability and random processes.

In this work we discuss two different methodologies: the classical
probabilistic approach, in which the properties are treated as random variables.
Stochastic Finite Element Methods are examined using both Monte Carlo
Simulation and Perturbation Methods. And the possibilistic approach,
represented by a model based on the theory of fuzzy sets.
THE MONTE CARLO METHOD

Introduction
In classical Monte Carlo Simulation N samples of the vector of random variables are generated from a specific joint probability density function. The implementation of the method consists in the numerical simulation of this samples associated to the random quantities of the physical problem. This method is completely general, for linear or nonlinear analysis, and have been used to calibrate and validate all other techniques. Since the accuracy of the statistical response, in sampling techniques depends on sample size, simulations can become very expensive requiring a great computational effort.

System Description
The distribution of the uncertainties of material properties is assumed to vary spatially as a two-dimensional homogeneous stochastic process and can be described as:

\[ A(x) = A_0 \left[ 1 + \alpha(x) \right] \]  

(1)

where \( A_0 \) is the mean value of the random property, \( \alpha(x) \) is the fluctuating component of the property and assumed to have mean zero. The autocorrelation function may be assumed as:

\[ R_{aa}(\xi) = \sigma^2 \exp \left[ -\left( \frac{\xi}{d} \right)^2 \right] \]  

(2)

where \( d \) is a correlation scale that governs the rate at which the autocovariance decays and \( \xi \) is the vector of the distance between two points.

The correlation characteristics can be specified in terms of the covariance matrix, whose generic component is given by:

\[ C_{ij} = \text{Cov} [\alpha_i, \alpha_j] = R_{aa}(\xi_{ij}) \]  

(3)

where \( \xi_{ij} \) is the distance between the centroid of element \( i \) and the centroid of element \( j \).

A vector \( \alpha = [\alpha_1 \alpha_2 \ldots \alpha_n]^T \) can be generated by \( \alpha = L \cdot Z \), where \( L \) is a lower triangular matrix obtained by the Cholesky decomposition of the covariance matrix \( C_{aa} \), and \( Z = [Z_1 \ Z_2 \ldots Z_n]^T \) is a vector of \( n \) uncorrelated Gaussian random variables with mean equal to zero and standard deviation equal to one.

The Direct Monte Carlo method uses the same formulation of the deterministic finite element method, realizing one analysis for each simulated sample after defining the stochastic field. Therefore, it is possible to get the statistics of the responses, having done the \( N \) simulations.
PROBABILISTIC FINITE ELEMENTS

In accordance with the idea of the perturbation approach to the stochastic version of the finite element method, the random variables may be expanded via Taylor series about their expected values, retaining up to second order terms. As a consequence of this truncation, this method is not very accurate when the variability of the properties increase. A major advantage is that only the first two statistical moments need to be known and not the multivariate distribution function. The random variables may be elastic moduli $D$, displacements $u$, or loads $F$. The expansions are done for a given small parameter $\kappa$, retaining terms up to second order. The expansion for $u$ are:

$$u[b(x_k); x_k] = u^0 + \kappa u^r \Delta b_r + \frac{1}{2} \kappa^2 u^{rr} \Delta b_r \Delta b_s$$

(4)

Where, $\kappa \Delta b_r = \delta b_r = \kappa [b_r(x_k) - b_r(x_k)]$

(5)

is the first order variation of the random field $b_r(x_k)$ about $b_r^0(x_k)$, $k=1,2,3$; and

$$\kappa^2 \Delta b_r \Delta b_s = \delta b_r \delta b_s = \kappa^2 [b_r(x_k) - b_r^0(x_k)] [b_s(x_k) - b_s^0(x_k)]$$

(6)

represents the 2nd order variation of $b_r(x_k)$ and $b_s(x_k)$

The notation $(.)^0$ indicates the expected value, whereas $(.)^r$ and $(.)^{rr}$ indicates the first and second derivatives with respect to the random field variables $b_r(x_k)$ evaluated at their expectations, respectively.

Adopting $b_r$ as a vector of random variables and $r$ varying from one to the total number of elements, the finite element equilibrium equations can be obtained substituting the expansions above into the deterministic finite element equilibrium equations. Separating the equations with equal order terms:

- Zeroth order ($\kappa^0$ terms, one system of equations)

$$K^0 u^0 = F^0$$

(7)

- First order ($\kappa^1$ terms, $n$ systems of equations)

$$K^0 u^r = F^r - K^r u^0$$

(8)

where $n$ is the number of random variables, and

$$K^r = \int_{V} B D^r B \, dV$$

(9)
\[ F^r = \int_V N f^r_V \, dV + \int_S f^r_S \, dS \]  

(10)

- **Second order (\( K^\alpha \) terms, one system of equations)**

\[ K^0 u^{(2)} = [F^{rs} - 2 K^r u^\alpha - K^{rs} u^\alpha] S_{b}^{rs} \]  

(11)

where, \( u^{(2)} = \frac{1}{2} \sum_{r,s=1}^{n} u^{rs} S_{b}^{rs} \)  

(12)

\[ K^{rs} = \int_V B D^{rs} B \, dV \]  

(13)

\[ F^{rs} = \int_V N f^{rs}_V \, dV + \int_S N f^{rs}_S \, dS \]  

(14)

and \( S_{b}^{rs} \) is the covariance between \( b_r \) and \( b_s \).

The equation of zeroth order is equal to the deterministic expression, thus it can be solved to get the displacements \( u^0 \). After that, the 1st order equation and the 2nd order equation can be solved.

In search of the probabilistic distributions of the displacements, the second order estimate of the mean value is obtained introducing the expanded value of the displacements equation (9) in the expression of the expected values:

\[ E[\{b(x_k); x_k^r\}] = u^0 + \frac{1}{2} u^{rs} (x_k) S_{b}^{rs} \]  

(16)

Substituting the second order expansions of the displacements in the expression of autocovariances, we arrive at the first-order accurate covariances of the displacements, which is compatible with the approach adopted here:

\[ S_{u} = u^r u^{rs} S_{b}^{rs} \]  

(17)

By the same way, the probabilistic expressions for the strains and stresses can be obtained.
THE POSSIBILISTIC APPROACH: FUZZY FINITE ELEMENT METHOD

Some uncertainties, specially those involving descriptive and linguistic variables, as well as those based on incomplete information are due to cognitive sources, they cannot be handled satisfactorily in the stochastic finite element method and the theory of fuzzy sets is more appropriate.

Fuzzy sets deal with the membership or non-membership of an object in a set with imprecise boundaries. The applicability of fuzzy set theory can also be extended to the situations which involve subjective uncertainty, or for which data are insufficient for statistical calculation. This theory has been extended to numerical methods for structural analysis, as the finite element method. In this method the computation of the element matrices requires the evaluation of integrals over the domain of the element. The global finite element equations are solved by well established techniques according the characteristics of the computational platform available. When we choose the model as a fuzzy system, the integrals of fuzzy quantities are to be evaluated over fuzzy domains and the methods of solving the global equations must consider that the information is fuzzy.

Construction of Membership Functions
The construction of membership functions is often possible if the expert knowledge exists. A membership function $\mu_A(x)$ denotes the grade of membership of element $x$ in the fuzzy subset $A$.

The membership functions of a fuzzy set can be also based on statistical data. In this case, it can be determined by approximating the probability density function shape or its estimated obtained from the histogram of the feature ($X$) considered for defining the fuzzy set.

Another alternative for the construction of membership functions based on statistical data, may be done from expected values and variances of a probability density function.

Triangular Membership Function
In structural engineering it is often possible to obtain knowledge about the values of parameters in the form of low, probable and high values. Based on this information, the membership functions can be constructed and the parameters are modeled as random variables based on the assumption that the properties are spatially variable. Following the concepts of the fuzzy set theory the parameters are modeled as fuzzy numbers in the finite element analysis, where the information about the values of these parameters in a specific situation is imprecise due to vague information.
In this work we have adopted the suggestion found in reference \(^{11}\) to define a linear membership function. A fuzzy parameter \(E\) (elastic modulus) can be represented in an interval of confidence as

\[
E_{\text{fuzzy}} = [E_L, E_R]
\]

where the subscripts \(L\) and \(R\) stand for the left and the right of a fuzzy number.

By constructing membership functions for the imprecise quantities, the fuzzy calculus and integration techniques are applied to obtain the finite element equations. A simplified methodology based on the permutation method \(^{12}\) for the analysis of the fuzzy parameters was adopted here. This alternative considers that in all the elements of the structural system, the extreme values are substituted to obtain the response extreme values.

**APPLICATIONS**

Consider a typical soil profile of a potash mine with a gallery subjected to a strip load and the mesh discretization shown in figure 1. The correlation length used is 3m and the load is 219000 kN/m.

![Figure 1](image_url)

The spatial variation and the autocorrelation function are defined as given by equation (1) and equation (4), respectively. The uncertain parameter considered is the elasticity modulus with an expected value equal to 1000000 kN/m\(^2\). The Poisson coefficient is equal to 0.2 and assumed as a deterministic property. In the fuzzy analysis a triangular membership was adopted. The results are presented in the points where the response is more significative, as the middle of the gallery span. The coefficients of variation of the random parameter are: 0.1, 0.2, 0.3 and 0.4.
In figures 2 and 3 the mean values and the standard deviation of the vertical displacements versus the coefficient of variation (COV) are plotted. In figure 4 the vertical displacements along the symmetry axes is plotted to a membership grade of 0.6. This value corresponds to a coefficient of variation equal to 20%. As expected the variation of the vertical displacements between the lower and upper bounds is wider for the case of fuzzy sets, but resulted in good approximation.

![Figure 2](image)

![Figure 3](image)
CONCLUSION

The treatment of uncertainties existing in engineering problems was traditionally done by the establishment of an upper bound on the maximum response of the system using the least favorable response. Nowadays, great effort is concentrated in this research area. The methods considered in this paper can be seen as practical tools for many real-life engineering problems. The development of high performance computers obviously contribute in this direction.

The extension of the methodologies for the solution of complex engineering analysis problems, including nonlinear applications, is currently under investigation.

REFERENCES


