Kinematic periodic structures (KPS) based on revolute hyperboloid
E.M. Kent
Faculty of Architecture, Technion City, Haifa, Israel

Abstract
It is possible to generate a rich variety of retractable structures by a novel application of two seemingly unrelated theories: The theory of kinematics, as it is pertained to to the behavior of kinematic one-sheet Revolute Hyperboloid (RH), and the theory of order and symmetry in space as it pertains to the analysis of the Elementary Periodic Region (EPR).

1 Introduction
Human imagination has always been stimulated by the kind of geometrical constructs called 'quadratic curved ruled surfaces'. These mathematical objects combine the tenderness of continuously varying curves with the staccato rhythm of straight lines, creating a notion which might be called 'frozen movement. Add to it the most wondrous property of all, The ability to collectively and continuously move and changes the appearance of the whole system; and the result will capture any ones imagination.

Mathematicians D.Hillbert and Cohen Vossen [1], had done important work dealing with these type of surfaces. In it they have proven that A Revolute Hyperboloid (RH) constructed of intersecting members which allow rotation only (scissor like) possess a global motion. The most general case for a Double Ruled Surface is a Single Sheet Elliptic Hyperboloid, of which the Single Sheet RH (fig.1) and the Parabolic Hyperboloid (PH) (fig.2) are special cases. Specific identification of such surfaces can be made according to the characteristic relationship between adjacent quadrangular segments in each surface (fig.3).
2 Tools & Terms

The following discussion will make use of tools borrowed from Kinematics and the Theory of Spatial Order, using EPR.

2.1 Basic Terms In Kinematics

The basic component of Kinematic is the Kinematic Chain (KC). Such KC is an assembly of rigid links of any shape. They are in turn, linked together in such a way as to permit but (one) Restrained Motion. Kinematic Chains, whether single looped or multi looped, possess usually but One Degree of Freedom (DOF); Special Geometrical Condition [2] may vary this number. The specific form in which two adjacent links relate to each other is denoted by their Kinematic Pair (KP). Table No. 1 shows some frequently used KP and their associated DOF numbers.

<table>
<thead>
<tr>
<th>Name of Pair</th>
<th>Abreviation</th>
<th>Symbol</th>
<th>Degrees of Freedom (DOF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revolute (Hinge)</td>
<td>R</td>
<td>![Link Symbol]</td>
<td>1</td>
</tr>
<tr>
<td>Prism (Slide)</td>
<td>P</td>
<td>![Link Symbol]</td>
<td>1</td>
</tr>
<tr>
<td>Helical (Screw)</td>
<td>H</td>
<td>![Link Symbol]</td>
<td>1</td>
</tr>
<tr>
<td>Cylindrical</td>
<td>C</td>
<td>![Link Symbol]</td>
<td>2</td>
</tr>
<tr>
<td>Totus</td>
<td>T</td>
<td>![Link Symbol]</td>
<td>2</td>
</tr>
<tr>
<td>Spherical</td>
<td>S</td>
<td>![Link Symbol]</td>
<td>3</td>
</tr>
<tr>
<td>Plane</td>
<td>P1</td>
<td>![Link Symbol]</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 1 Examples of Kinematic Pairs

2.2 The Basic Kinematic Unit of the RH

It is possible to consider the 5 Members Unit (5MU), described in fig.3, as the most RH elementary unit which allows motion. A surface having an infinite number of rectangular segments, may be dissected in such a way as to leave in the cut out section at least one 5MU. In such case, it will still possess the Restrained Movement of the whole system; which means that is has a DOF number of 1. The common criterions used to evaluate such Kinematic system are those developed by Grabler (2-D, Eqn.1) and
\[ F = 3(n-1) - 2P_1 - P_2 \]  
(1)  
\[ F = 6(n-1) - 5P_1 - 4P_2 - 3P_3 - 2P_4 - P_5 \]  
(2)  

\( F \) - Sum total of Degrees of freedom (DOF) in the kinematic configuration  

\( n \) - Number of members in the kinematic configuration.  

\( P(i) \) - Number of kinematic couplings in the kinematic configuration, allowing for \( i \) DOF.  

Analysis of the elementary Unit of the RH according to Kuetzbach:  

All connections are considered to be Spherical KP (S), (fig.4) each possessing 3 degrees of freedom, hence:  

\[ F = 6 \times (5-1) - 3 \times 6 = 6 \]  

While each of the members in the system may rotate axially, such motion is proven to be globally irrelevant and therefore can be overlooked in our analysis. Thus we are left with a simple kinematic system whose restrained motion is allowed by one degree of freedom.

2.3 Theory of order in space

Periodical Configuration (PC), is a group of elements arranged in \( n \)-dimensional space in any periodical order. A PC possesses one or more elements of Symmetry; It may also include a Symmetry Group of a higher order - such as exists in periodical finite and infinite Polyhedra [3], as well as in regular periodical tessellations. Using Coxeter’s terminology, those might be noted as 2.3.6, 3.3.3, 2.4.4, 2.2.2.2. respectively (Fig.5). In the following discussion we will come back and refer to them.

<table>
<thead>
<tr>
<th>2.3.6</th>
<th>3.3.3</th>
<th>2.4.4</th>
<th>2.2.2.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>EPR pattern</td>
<td>EPR pattern</td>
<td>EPR pattern</td>
<td>EPR pattern</td>
</tr>
<tr>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
<td><img src="image4.png" alt="Image" /></td>
</tr>
</tbody>
</table>

Figure 5: The four symmetry groups used.
Subdivision of a periodical configuration by means of its elements of symmetry (reflection planes, reflection axes, etc.), yields the basic symmetrical module – the "atom", that which can not be further subdivided; It will, at the same time, subdivide the containing n-dimensional space into fundamental regions. These "atoms" are called the Elementary Periodic Regions (EPR). The elements of symmetry which define the indivisible portion of the periodic configuration define also the envelope of the EPR and is an integral part of it. Inherent in this association is the concurrent propagation of the both special as well as spatial periodic orders.

3 Discussion

The following discussion deals with the properties of the Geometric Invariant of the RH and its application, in order to combine (join) surface patches (segments) into larger kinematic systems. We will further discuss RH segments having symmetrical characteristic and their associated symmetry groups. Finally we will show the application of the EPR’s in symmetry operations, and demonstrate its use as a generator of various kinematic infinite periodic formations.

3.1 The Geometric invariant of the RH Surface

In a research carried out by the author, a surprising geometrical fact had been observed, which might serve as just such a generator: Let the RH move within the constraints of its restraining motion, and let us observe its projected image on its projection Plan Parallel (PP) to it's main axis (fig.6); then we would observe that the any change in the size of such image corresponds directly to the movement of the RH, while at the same time preserving all of its proportions. Moreover, any information marked on the surface will similarly change it's dimensions (fig.7) [4]. We may relate to this property as to a Geometrical invariant in a varying geometry.

![Figure 6: Parallel Projections of two phases of RH](image1)

![Figure 7: Projected shape](image2)
This procedure can also be reversed. For instance, by projecting a closed polygon, marked on the projection plane back to the RH surface, we can obtain a closed form with 3-D curved edges (fig.8). If we then segment (cookie cut) such a form, it will preserve its restraint motion as long as it will still contain the elementary unit of motion – the SMU described in fig.3. This does not hold true in cases where the system operate under edge conditions.

**3.2 RH segments with symmetry attribute**

The projection envelope of a closed polygon, as defined by its correspondence between 2D and 3D surfaces, is a right angle prism. In this context it might be termed The Slicing Prism. In cases where a closed shape projected onto the RH is of a regular symmetry, this attribute will be preserved in the Slicing Prism as well. Further more, in situations where these derived shapes allow for periodic tessellations of 2D space – the Slicing prism would constitute an EPR!. Since the projection operation is complementary in nature, it can be shown that a kinematic change in the size of surface segment causing in turn a change in the projected image, would also effect a corresponding change in the Slicing Prism, i.e. changing its size but preserving all of its symmetry attributes, including those allowing for regular tessellations.

This phenomena, in which a kinematic segment of an RH is projected through a plane surface, and this same surface is both the slicing plane as well as the projection plane is not an obvious one. Combining any two kinematic surface segments, each having a restrained motion, into one one kinematic system having one restrained motion, is possible only under the following conditions (illustrated in fig.9):

1. There should be a perfect match between curves A and B, derived from cutting the segments.

2. There ought to be an exact identity in the behavior of the adjoining edges, as defined by their kinematic change.
It follows then, that since all symmetry and other formal attributes are being preserved in the kinematic transformation, the plane on which the curve joining the two adjoining surfaces lies, will also be preserved, and thus all 2D tessellations would be preserved as well!

It should be noted however, that the effect of connecting the two projected segments at many connecting points along the mutual edge instead of the more common 3 type S kinematic pairs, will generally bring the numerical analysis to yield a large negative DOF value, while in actuality the system still possesses movement capability.

### 3.3 Combining HS segments into periodic system

Using four symmetry groups based on EPR and their projection planes, it is possible to form HR segments into all 2-D periodic systems. By defining the relationship between the EPR and the RH - in both size and location, as demonstrated in fig. 10, we define the periodic atom and correspondingly periodic system as a whole.

Note that Symmetry group (SG) 3.3.3 can be seen as a sub group to SG 2.3.6, and the same applies to SG 2.2.2.2 and SG 2.4.4 (fig.11)
Table 2 describes a partial set of periodic systems, their EPR, RH, their Slicing Prism and their generating mechanism.
4 A Case Study

The project described in the following section was presented as a B.Arch. final project at the Technion – Technological Inst. of Israel, by Mr. R. Bar under the Author’s direction. It is used here in order to introduce a chosen list of relating topics. These are general topics, relevant to any project applying the principles of kinematic periodic ruled surfaces in structural folded systems.

The project deals with the design of a 1200 sqm folded roof for a flower exhibit: fig.12 shows the original RH EPR and the specific Slicing Prism. fig.13 shows the vertical projection of a group of EPR. fig.14,15 show typical section and perspective views at a folded state.

Figure 12: EPR and HR
Figure 13: Projectio and view of one wing
4.1 The definition of sub systems in the overall form by means of their general movement

Any kinematic change in the participating forms, entail a dramatic change in their size. This in turn calls for a physical 'translation' of their supports, as shown in fig.16 and 17.
4.2 Deformation in the system

The process of translating the theory into practice; the geometry into a physical entity, brings with it an inevitable deformation: Physical members inhabit 3D space and can no longer behave as 1D vectors for example, thus requiring joints design which take into account their non-merging line of action. It follows obviously that the system can no longer be folded to its theoretical limit.

4.3 3-D motion of joints - Design considerations

The relative motion of two connected members is proportional to their distance from the RH 'hips'. It combines two rotational movements: one is a scissors like motion around the joint, while the other is a rotation of each member about its axis. This complex action should also be accommodated in the designing of a joint.

4.4 Covering material - Design considerations

The geometry of the covering material is shown to also undergo a major change in actual system operation. It seems therefore that the most logical material to use in this case is a fabric. In designing the membrane, the inevitable deformation in the fabric have been taken into account in such a way as to use a pre-tensioning effect occurring as the structure is unfolding. Fig.18 shows a periodic segment of the fabric.

4.5 System integration

The translation of kinematic geometry into physical structure is a process which envolve a rather complex aggregate of interdependent systems, together with their associated kinematic action. In order for such a system to function in concert, all of its components motion should be coordinated into a global restrained motion of one. This in turn can be achieved only through a meticulous consideration of its parts: links; bars and the like. The design ought to refer constantly back to the system geometry, while at the same time, a careful reciprocal attention should be given to the 'modified geometries' of real 3D life.
5 Conclusion

It seems that the application of RH systems may yield a rich variety of solution for the purpose of designing retractable and folding structures. The research into use of the EPR and RH as a 'form generator' is still young, but already shows a great promise.

6 References


