The effect of wall slip on the migration rate of concentrated suspensions in a Couette device

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Abstract

A rheological model for the flow of concentrated suspensions based on the volume-averaged velocity of suspended particles in a viscous fluid has been developed. The so-called diffusive-flux model is employed to investigate the demixing rate of neutrally-buoyant suspensions in a wide gap Couette device. The model employs two tuning parameters (diffusion coefficients) $K_c$ and $K_\eta$ which govern the relative rates and magnitudes of particle diffusion caused by gradients in the particle interaction frequency and viscosity, respectively. These tuning parameters were originally sized based on no-slip boundary conditions. However, because the velocity is volume-averaged over the fluid and solid phases, slip does exist at the walls. In this study, the effect of slip on the solid-phase migration rate is investigated. The governing model equations are solved using a finite difference method for various assumed values of the slip coefficient. In general, it is found that slip retards the migration rate.

1 Introduction

Initially well mixed neutrally-buoyant particles suspended in nonlinear shear flows have been shown to undergo migration from high shear rate regions to low shear rate regions. More specifically in a Couette device, the particles tend to migrate from the inner rotating cylinder wall to outer stationary cylinder wall [1, 3, 5]. A phenomenological model to describe the migra-
tion process has been developed by Phillips et al. [9] and Subia et al. [10]. The so-called diffusive-flux model couples the volume-averaged suspension momentum equation with an advection-diffusion equation for the evolution of the solid-phase concentration. The diffusive-flux model contains two tuning parameters, $K_c$ and $K_\eta$, which represent the diffusion coefficients associated with "collision-induced" and "viscosity-induced" migration, respectively. The diffusive-flux model was improved by Fang et al. [4] to allow for nonisotropic migration. In particular, a flow aligned-tensor implementation of the diffusive-flux model was developed to allow for different migration rates within and perpendicular to the shear plane.

The models and their modifications have been shown to yield satisfactory results for the steady-state concentration profiles when compared to experimental results [11, 8, 4]. Unfortunately, the comparison of the transient results between model and experiment have shown large discrepancies in several cases. In particular, the diffusive-flux model predicts that the migration rate should scale with the representative particle radius squared. However, experimental results at a bulk concentration of 50% indicate that the migration scales with the particle radius to somewhere between the powers of 2.5 to 2.9 [1, 11].

In the current diffusive-flux model, the diffusion coefficients $K_c$ and $K_\eta$ are functions of the local solid-phase concentration $\phi$ only. The model results can be made to match experimental results far better if these coefficients are allowed to be functions of both the local concentration, $\phi$, and the relative particle size, $\epsilon = a/R_\alpha$, where $a$ is the characteristic particle radius and $R_\alpha$ is the outer radius of the Couette device. However, in order to determine the proper functional relationship of the diffusion coefficients on $\phi$ and $\epsilon$, the appropriate boundary conditions at the wall must first be determined. In this research, the effect of wall slip on the migration rate of concentrated suspensions in a Couette device is investigated.

Previous implementations of the diffusive-flux model have assumed no slip boundary conditions at solid walls for the suspension velocity [10, 11]. However, since the suspension velocity is volume averaged over both the fluid and solid phases, slip boundary conditions will exist at the wall since the solid particles can roll along the wall. A recent investigation [7] indicated that the slip will increase with increasing particle size and solid-phase concentration so that the slip boundary condition will also depend on $\phi$ and $\epsilon$.

This paper investigates the effect of wall slip on migration characteristics in a wide-gap Couette device. A slip boundary condition is imposed for the diffusive-flux model. The equations are solved numerically using a finite difference method. The effect of the slip parameter on the development of the concentration profile is investigated.
2 Governing equations

2.1 Modified diffusive-flux model

The multiphase systems considered here are solid particles suspended by a Newtonian fluid. For neutrally-buoyant particles, the balance equations for an incompressible suspension are given by

\[ \nabla \cdot \mathbf{u}_s = 0 \]  

\[ \frac{D \rho \mathbf{u}_s}{Dt} = \mathbf{b} + \nabla \cdot \mathbf{\sigma} \]  

where

\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_s \cdot \nabla \]  

is the substantial derivative and \( \mathbf{u}_s \) is the suspension velocity. The suspension stress tensor \( \mathbf{\sigma} \) is given by

\[ \mathbf{\sigma} = -p \mathbf{I} + 2\eta(\phi)\mathbf{D} - \eta_p \dot{\gamma} \mathbf{Z} \]  

where \( p \) is the suspension pressure, \( \eta \) is the effective suspension viscosity, \( \mathbf{D} \) is the suspension deformation rate tensor, \( \mathbf{I} \) is the identity matrix, \( \dot{\gamma} \) is the shear rate, and \( \mathbf{Z} \) is the flow-aligned tensor discussed below.

The effective suspension viscosity is modeled using the Krieger correlation [10]. Hence, \( \eta = \eta_0 \eta_r \) where \( \eta_0 \) is the solvent viscosity and \( \eta_r \) is the relative viscosity for the suspension given by

\[ \eta_r = \left(1 - \frac{\phi}{\phi_m}\right)^{-\alpha} \]  

where \( \phi_m \) is the maximum solid volume fraction for which the suspension exhibits fluid behavior. The value of \( \phi_m \) depends on several factors as discussed by Subia et al. [10]. In this research, \( \phi_m \) is chosen to be 0.68 and \( \alpha \) to be 1.82 in all calculations.

The migration of particles is governed by the following advection-diffusion equation

\[ \frac{D \phi}{Dt} = -\nabla \cdot \mathbf{N} \]  

where \( \mathbf{N} \) has two contributions, one due to interparticle hydrodynamic interactions, \( \mathbf{N}_c \), and one due to spatial variations in viscosity, \( \mathbf{N}_\eta \).

The two particle flux terms are then modeled as [4]

\[ \mathbf{N}_c = -K_c a^2 \phi \nabla \cdot (\dot{\gamma} \phi \mathbf{Z}) \]  

and

\[ \mathbf{N}_\eta = -K_\eta a^2 \phi (\dot{\gamma} \phi \mathbf{Z}) \cdot \nabla \ln \eta \]
The flow aligned tensor, \(Z\), is given by

\[
Z = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}
\] (9)

The flow aligned tensor is diagonal only in the so-called shear-axis coordinate system defined by \(\delta_1\), \(\delta_2\), and \(\delta_3\) [2]. The shear axes are based on the shear surfaces moving with the ambient flow. At any instance of time, \(\delta_1\) and \(\delta_3\) are tangent to the shearing surface. In particular, \(\delta_1\) is in the direction of the streamline and \(\delta_3\) is in the direction of the vortex line. Hence, \(\delta_2\) is normal to the shearing surface in the direction of the velocity gradient. Based on comparisons with experimental results for suspension flows in a parallel plate rheometer and a cone-and-plate rheometer, the following relation can be shown between the parameters [4]

\[
\lambda_1 = \lambda_2 = 2\lambda_3
\] (10)

The flow aligned tensor accounts for the nonisotropic nature of the migration within and out of the shear plane.

In the circular Couette flow geometry, the diffusive flux equation (Eq. 6) reduces to

\[
\frac{\partial \phi}{\partial t} = \frac{a^2}{r} \frac{\partial}{\partial r} \left\{ r \left[ K_c \left( \phi^2 \frac{\partial \gamma}{\partial r} + \phi \frac{\partial^2 \phi}{\partial r^2} \right) + K_n (\gamma \phi^2) \frac{1}{\eta} \frac{d\eta}{d\phi} \frac{\partial \phi}{\partial r} \right] \right\}
\] (11)

### 2.2 Slip velocity boundary condition

The slip velocity boundary condition is given by

\[
\beta a \gamma = (u_p - u_f) \cdot t
\] (12)

\(t\) is the unit tangent vector to the boundary in the direction of slip, \(u_p\) and \(u_f\) are particle phase and fluid phase velocities, respectively, and \(\beta\) is slip coefficient. As discussed above, the slip coefficient \(\beta\) is a function of the relative particle size, \(\epsilon\), and the local particle concentration, \(\phi\). At the two extremes, when \(\beta = 0\), there is no slip at the wall and when \(\beta = \infty\), there is perfect slip at the wall.

The suspension phase velocity is given by

\[
u_s = (1 - \phi)u_f + \phi u_p
\] (13)

in which \(\phi\) is the local particle concentration. Hence, inserting Eq. 12 into Eq. 13, the slip boundary condition in the circumferential direction is given by

\[
u_s^\theta = u_f^\theta + a\beta \gamma \phi
\] (14)

where \(u_s^\theta\) and \(u_f^\theta\) are the \(\theta\)-components of the suspension and fluid velocities, respectively.
Using the outer cylinder radius, $R_o$, as the length scale, $\omega R_o$ as the velocity scale where $\omega$ is the angular velocity of the inner cylinder, and $1/(\epsilon^2 \omega)$ as the time scale, the dimensionless governing equations take the form

$$ \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \eta \frac{\partial}{\partial r} \left( \frac{u_s^\theta}{r} \right) \right] = 0 $$

(15)

$$ \frac{\partial \phi}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left\{ r \left[ K_c \left( \phi^2 \frac{\partial \gamma}{\partial r} + \phi \frac{\partial \phi}{\partial r} \right) + K_\eta (\gamma \phi^2) \frac{1}{\eta} \frac{d\eta}{d\phi} \frac{\partial \phi}{\partial r} \right] \right\} $$

(16)

The slip velocity boundary conditions are given by

$$ u_s^\theta |_{r=r_i} = 1 - \phi e \beta \gamma |_{r=r_i} $$

(17)

$$ u_s^\theta |_{r=r_o} = \phi e \beta \gamma |_{r=r_o} $$

(18)

and the no-flux boundary condition for concentration is given by

$$ N_r = K_c \left( \phi^2 \frac{\partial \gamma}{\partial r} + \phi \frac{\partial \phi}{\partial r} \right) + K_\eta (\gamma \phi^2) \frac{1}{\eta} \frac{d\eta}{d\phi} \frac{\partial \phi}{\partial r} = 0 $$

(19)

As discussed above, in general, the slip coefficient $\beta$ is a function of both the relative particle size, $\epsilon$, and the local particle concentration, $\phi$. A set of experiments [6] performed in a Couette device showed that the dependence of $\beta$ on $\phi$ could be correlated to the effective suspension viscosity through the equation

$$ \beta = C_\beta \eta_r $$

(20)

where $C_\beta$ is a proportionality constant. This model for the slip coefficient is used in the subsequent analysis of the next section.

3 Analysis and numerical results

In this paper, a linear model is chosen for the ratio of the diffusion coefficients given by $K_c/K_\eta = 1.2 \phi$. The governing system (Eqs. 15-19) is solved using a finite difference method. In particular, the shear rate $\dot{\gamma}$ is first obtained by integrating Eq. 15. Next, an implicit time and central difference in space scheme is used to discretize Eq. 16. A typical non-dimensional time step is on the order of $10^{-4}$.

For the linear approximation of $K_c/K_\eta$, an analytic solution exists for the steady state concentration profile given by

$$ \left( \frac{r}{R_o} \right)^2 = \left( \frac{\phi}{\phi_o} \right)^{\left( \frac{1}{1+i^2 \phi_m} \right)} \left( \frac{\phi_m - \phi_o}{\phi_m - \phi} \right)^{\alpha \left( \frac{1}{1+i^2 \phi_m} - 1 \right)} $$

(21)

where $\phi_o$ is the local solid-phase concentration at the outer wall. It is apparent that the steady state concentration profile is uniquely determined by the ratio of $K_\eta/K_c$ and is not affected by the slip coefficient, $\beta$. A
comparison of the steady state results from this model with experiment [11] is shown in Figure 1. The match between the steady state model and the experimental results is seen to be quite good.

To determine the effect of the slip coefficient $\beta$ on the transient solid phase concentration profiles, a difference measure, $D^s$, is defined in an areal weighted manner by

$$D^s = \frac{1}{A} \int_{r_i}^{r_o} |\phi^t - \phi^s| r dr$$  \hspace{1cm} (22)

where $\phi^s$ is the steady state concentration profile given by Eq. 21, $\phi^t$ is the transient concentration profile at time $t$, $r$ is the radial position and $A$ is the total cross-sectional area of the Couette device. Hence, the difference measure essentially measures how far the transient concentration profile is from steady state.

The difference measure for a variety of values of the coefficient $C\beta$ is
shown in Figure 2. These simulations were run from an initial uniform concentration profile of 50%. As seen in the Figure, the rate of particle migration decreases with increasing slip. The slip on the wall retards the formation of microstructure within the suspension which results in a longer time to run through the transient to achieve steady state.

The evolution of the slip velocity at the inner and outer cylinders as defined by Eqs. 17 and 18, respectively, can be calculated using the numerical simulation. A plot of these slip velocities as a function of time is shown in Figure 3. As seen in the Figure, the slip velocity decreases substantially on the inner cylinder because of the outward migration of particles. Interestingly enough, it appears that the slip velocity at the outer cylinder actually decreases slightly over time even though the particle concentration is increasing in the vicinity of the outer cylinder. The reason for this is that the local shear rate is decreasing near the outer cylinder reducing the slip.

Assuming that the diffusion coefficients $K_c$ and $K_p$ do not depend on the relative particle size $\epsilon$, the migration rate of particles would scale as $\epsilon^2$ without taking into account for velocity slip at the wall. Experimental evidence indicates that for bulk concentrations of 50%, the migration rate
Figure 3: Slip velocity at the inner (solid curve) and outer (dashed curve) cylinder walls for $C_B = 1.0$, $\epsilon = 0.0589$, and a bulk solid-phase concentration of 50%.

of the particles scales approximately as $\epsilon^{2.7}$ [11]. Hence, the experimental results point to a deficiency in the model. To show the effects of slip on the migration rate, a scaled time is defined as $t \omega K \epsilon^\eta$. For various values of the coefficient $C_B$, the exponent $n$ is adjusted until the difference measures as a function of the scaled time overlays for two different $\epsilon$'s, namely, $\epsilon = 0.0266$ and $\epsilon = 0.00589$. Results are shown in Figure 4. As seen in the Figure, the exponent $n$ must be decreased below 2 as $C_B$ is increased in order for the difference measure for different $\epsilon$'s to have the same functional dependence on the scaled time. Unfortunately, this is the opposite trend as shown in experiment where $n$ was shown to be approximately 2.7 for a 50% bulk concentration suspension.
Figure 4: The impact of slip on the migration rate. All difference measures for different size particles are collapsed into a single curve when time is scaled as $K_e \omega t e^n$. $n = 1.92$ for $C_\beta = 1/4$ (top), $n = 1.85$ for $C_\beta = 1/2$ (middle), and $n = 1.75$ for $C_\beta = 1$ (bottom).
4 Conclusions

The effect of velocity slip on the migration characteristics of concentrated suspensions has been studied. Initial investigations have shown that the general effect of slip is to decrease the migration rate. In general, slip is a function of the relative particle size \( \epsilon \) and the local concentration \( \phi \). Although Eq. 20 gives a functional dependence of the slip coefficient \( \beta \) on the local concentration \( \phi \), it says nothing about the dependence of \( \beta \) on \( \epsilon \). The current results also show that slip actually reduces the exponential scaling of the relative particle size on the migration rate in contrast to experiment. This research has shown that, before a complete theory for particle migration in concentrated suspensions can be developed, velocity slip at the wall must be characterized. Additional experiments are required to determine the functional relationship of slip on the relative particle size and the local solid phase concentration. Only after this relationship has been determined can reliable data concerning the diffusion coefficients \( K_c \) and \( K_H \) be generated.

5 Acknowledgement

This work was partially supported by the United States Department of Energy (DOE) grants DE-FG03-97ER14778 and DE-FG03-97ER25332. This financial support does not constitute an endorsement by the DOE of the views expressed in this paper.

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