A numerical solution to the suspension thickening problem

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Abstract

The process of a transient sedimentation and the subsequent gravity thickening of a suspension is investigated. Making use of the partial differential equations describing the Darcian mechanics of two-phase systems, the process is formulated mathematically. The equation for the motion of the interface between the compression zone and the sediment zone is derived to complete the governing system of equations. The resulting non-linear moving boundary problem is solved numerically and the obtained results are presented.

1 Introduction.

The process of compression of a suspension has to be known in detail when various industrial technologies are designed, e.g. when the optimum height of the compression zone of sedimentation tanks has to be determined. According to its energetic efficiency, the gravity thickening, solved in this paper, is one of the most convenient processes of lowering the volume of suspensions and significantly increasing the concentration of their solid phase. While the problem of sedimentation zone and the problem of cross-section area of the thickening tank were solved, e.g. Handová and Sladká (1988), Koníček and Burdych (1988), and Tuček and Koníček (1989), the problem of finding the optimum height of the compression zone is solved by
means of semi-empirical methods. In this paper, the idea suggested by Mls, Koníček and Handová (1992) is followed and a problem of sedimentation and subsequent gravity compression of a suspension is solved numerically.

2 Governing equations

Mls (1995) presents the following set of partial differential equations describing the Darcian mechanics of isothermal processes in suspensions. For the case of one-dimensional process with x-axis oriented vertically upwards, the equations read

\[
\frac{\partial n}{\partial t}(x,t) + \frac{\partial w}{\partial x}(x,t) = 0 ,
\]

\[
\frac{\partial n}{\partial t}(x,t) - \frac{\partial v}{\partial x}(x,t) = 0 ,
\]

\[
\frac{\partial w}{\partial t}(x,t) + gn(x,t)(\frac{u(x,t)}{k(n)} + \frac{\partial H_w}{\partial x}(x,t)) = 0 ,
\]

\[
\frac{\partial v}{\partial t}(x,t) - g \frac{n(x,t) u(x,t) \rho_w}{\rho_s k(n)} + (1-n(x,t)) g \frac{\partial H_s}{\partial x}(x,t) - \frac{1}{\rho_s} \frac{\partial \tau}{\partial x}(n) = 0 ,
\]

where \( t \) is time, \( n \) is porosity, \( w \) and \( v \) are (volumetric) flux density vectors of the liquid phase and the solid phase respectively, \( u \) is the relative flux density of the liquid phase satisfying

\[
u = w - \frac{n}{1-n} v ,
\]

\( g \) is gravitational acceleration, \( k \) is hydraulic conductivity, \( \rho_w \) and \( \rho_s \) are (here constant) liquid and solid phase densities, respectively, \( H_w \) is hydraulic head, \( \tau \) is effective stress of the solid phase, and

\[
H_s = x + \frac{p}{\rho_s g} ,
\]

where \( p \) is the liquid phase pressure.

3 The solved problem

The problem to be solved is a transient sedimentation and subsequent compression of a vertical suspension column. The initial height of the
column will be denoted \( L \). The initial distribution of the suspension in the column is homogeneous with zero velocities, i.e.

\[
c(x, 0) = c_0, \\
w(x, 0) = 0, \\
and \qquad r(x, 0) = 0,
\]

where \( x \in (0, L) \) and

\[
c(x, t) = \rho_s (1 - n(x, t))
\]

is the concentration of the solid phase in the suspension and \( c_0 \) is a constant value. It is supposed that the initial concentration \( c_0 \) is sufficiently small to prevent mutual interactions between individual particles of the solid phase. Consequently, the initial value of the effective solid-phase stress is zero,

\[
\tau(x, 0) = 0, \quad x \in (0, L).
\]

Let us further suppose that the bottom of the column, i.e. its lower edge \( x = 0 \), is impervious for both the phases. Hence, two boundary conditions follow

\[
w(0, t) = 0, \\
and \qquad r(0, t) = 0.
\]

for \( t > 0 \).

The aim is to investigate the process of the redistribution of the suspension, the motion of its phases and changes of pressure, effective stress and concentration, i.e. to find functions \( w, r, p, \tau \) and \( c \) within the column and a time interval \((0, T)\).

### 4 Constitutive equations

The system of governing equations (1) to (4), which are nothing but two continuity equations and two equations of motion, contains six unknown functions \( w, v, n, p, \tau \) and \( k \) of independent variables \( x \) and \( t \). Consequently, it is necessary to add two additional functions in order to complete
it in the form of a closed system. These two equations will characterize
the particular suspension under consideration.
We will suppose that the hydraulic conductivity \( k \) is a function of porosity.
Moreover, since the investigated process evidently satisfies
\[
\frac{\partial n}{\partial t}(m,t) \leq 0
\]
for every value of a material coordinate \( m \) and for every time \( t \), it is a
monotonous process which is defined, according to Mls (1995), by the
condition that the inequality
\[
\frac{\partial n}{\partial t}(m_1,t_1) \frac{\partial n}{\partial t}(m_2,t_2) \geq 0
\]
is satisfied for every two values \( m_1, m_2 \) and every two values \( t_1, t_2 \). Hence,
the hysteresis effect can be excluded and it can be assumed that the ef-
fective stress is a monotonous differentiable function of porosity.
Making use of a set of experiments made with a suspension consisting of
a mixture of kaolin and water and for processes of that kind described in
this paper, following functions were derived by Mls (1995)
\[
k(c) = A c^B , \quad (11)
\]
and
\[
\tau(c) = \begin{cases} 0 & \text{for } c \leq a + b , \\ g \frac{\rho_w - \rho_s}{\rho_s} \left( c + a \ln \left( \frac{c - a}{b} \right) - a - b \right) & \text{for } c > a + b . \end{cases} \quad (12)
\]
where \( A = 18.6, B = -2.468, a = -67.206 \) and \( b = 151.55 \) if \([k] = \text{m/s}, [\tau] = \text{Pa}\) and \([c] = \text{kg/m}^3\). Equations (11) and (12) are constitutive equations
added to the solved system of equations.

5 Moving boundaries

Two interfaces develop in the suspension column during the considered
process. The upper one separates the zone of sedimenting suspension
from the overlying layer of water. It starts from the top of the column
and, at every time, it makes the upper boundary of the solid-phase body.
of the suspension. Denoting $Z(t)$ its height above the bottom at time $t$, it holds

$$Z(0) = L$$

and

$$\int_0^{Z(t)} c(x, t) \, dx = \rho_s (1 - n_0) L.$$

The lower interface separates the sedimentation zone from the zone of compression which develops at the bottom of the column as a result of the imperviousness of the bottom. The mutual interactions of solid-phase particles cause negative values of the effective solid-phase stress within the compression zone. Hence, the height $Y(t)$ of the lower interface above the bottom, i.e. the height of the compression zone at time $t$, is defined by equation

$$Y(t) = \sup\{x; \tau(x, t) < 0\}. \quad (13)$$

As the value of the effective stress is zero in the zone of sedimentation, the governing equations are rather simplified for $x$ between $Y(t)$ and $Z(t)$. Hence, the considered problem will be solved separately for the sedimentation zone and for the compression zone.

6 The zone of sedimentation

Denote

$$\Omega_s = \{(x, t); \ t > 0, \ x \in (Y(t), Z(t))\},$$

and introduce

$$\epsilon(x, t) = \frac{n(x, t)}{1 - n(x, t)},$$

where $\epsilon$ is called the void ratio.

The initial homogeneity of the suspension, Eq. (5), and the absence of the effective stress in $\Omega_s$ make it possible to find following solution

$$n(x, t) = n_0$$

and

$$w(x, t) = k_0 \frac{\rho_s - \rho_w}{\rho_w} (1 - n_0)^2 \left(1 - \exp\left(\frac{-g \rho_w \epsilon_0 t}{k_0 (n_0 (\rho_s - \rho_w) + \rho_w)}\right)\right).$$

where $k_0 = k(c_0)$ and $(x, t) \in \Omega_s$. 
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Because of the absence of solid-phase particles above \(Z(t)\), it holds

\[
\frac{dZ}{dt}(t) = -\frac{w(Z(t), t)}{1 - n_0}
\]

and consequently

\[
Z(t) = L - (\rho_s - \rho_w)(1 - n_0)\left[\frac{k_0 t}{\rho_w} + \frac{\left(n_0(\rho_s - \rho_w) + \rho_w\right)}{g\epsilon\rho_w^2}k_0^2\left(1 - \exp\left(-\frac{g\rho_w\epsilon_0 t}{k_0(n_0(\rho_s - \rho_w) + \rho_w)}\right)\right)\right].
\]

7 The zone of compression

Denote

\[\Omega_c = \{(x, t); \ t > 0, \ x \in (0, Y(t))\} .\]

The problem of the solid-phase motion and the liquid-phase flow in the zone of compression is defined by the governing equations (1) to (4) and by the constitutive equations (11) and (12). The prescribed initial conditions are (5), (6), (7) and (8), and the imposed boundary conditions are (9), (10) and

\[\tau(Y(t), t) = 0 , \quad (14)\]

for \(t > 0\), where the last condition results from the definition (13) of the interface between the compression zone and the sedimentation zone. (of the upper boundary of the compression zone). Making use of the constitutive equation (12), it is possible to replace condition (14) by the condition

\[c(Y(t), t) = a + b\]

The imposed boundary conditions would be sufficient in the case of the known boundary \(\partial\Omega_c\) of the domain \(\Omega_c\). As the position of the lower interface is defined by the unknown function \(Y(t)\), it is necessary to add one more condition for the upper boundary of the compression zone.

From the condition of mass continuity, following equation can be derived

\[
\frac{dY}{dt}(t) (n_+(t) - n_-(t)) = w_+(t) - w_-(t) . \quad (15)
\]

where

\[w_+(t) = \lim_{x - Y(t)^+} w(x, t)\]
and

\[ w_-(t) = \lim_{x \to Y(t)^-} w(x, t) \]

and similarly for functions \( n_+ \) and \( n_- \).

Adding Eq. (15) to those conditions introduced above, the mathematical formulation of the problem is completed. Based on the described formulation, the process under consideration was solved numerically.

The initial height of the column was \( L = 1 \) m and the initial concentration was \( c_0 = 26.82 \) kg/m\(^3\).

Figure 1: Function \( Y(t) \)

Figure 1 presents the resulting shape of the function \( Y(t) \). After approximately 150 minutes of sedimentation, the interfaces meet, the sedimentation zone disappears and the compression zone reaches its maximum
height. The process of compression continues with decreasing velocity until the steady state is reached.

Figure 2: Concentration profiles of the compression zone

Two functions presenting distribution of concentration in the compression zone are shown in Figure 2. The triangles show the concentration profile at the time when the interfaces meet, the squares show the concentration profile at the steady state. It is the maximum concentration reachable by means of gravity thickening.

8 References


