Dual reciprocity Boundary Element Method formulation of coupled energy, mass and momentum transport in solid-liquid phase change systems

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Abstract

The problem of time dependent two-dimensional coupled energy, mass, and momentum transport in solid-liquid phase change systems has been formulated for the dual reciprocity Boundary Element Method solution procedure. The physical model is based on one-phase formulation which incorporates the connected and disconnected solid phase limits, and distinct or continuous phase change. Provisions for non-linear boundary conditions and temperature variation of all material properties are included. The related energy equation is source-term formulated, Kirchhoff transformed, and comprises the convective-diffusive extension of the Voller-Swaminathan timestep iteration strategy. The related Navier-Stokes type mass and momentum equations are formulated in primitive variables, and include the hypersingular consistent pressure gradient Poisson equation. The numerical solution of all equations involved is based on Laplace equation fundamental solution weighting and scaled augmented thin plate splines for transforming the domain integrals into a finite series of boundary integrals.

1 Introduction

Understanding transport phenomena in solid-liquid phase change systems is of fundamental importance in basic and applied sciences. An overview of the discipline could be found in [1]. Evaluation of the relevant physical models requires the use of the discrete approximative solutions since correlations and analytical solutions are rare in this highly non-linear field. An extensive survey on the use of the boundary element method in solid-liquid phase change systems context is provided in [2]. Very recently, extensive progress has been made in solving convective-diffusive solid-liquid phase
Moving Boundaries IV

change problems by the dual reciprocity method [3]. The method is now able to cope simultaneously phase-change, non-linear material properties, and boundary conditions [4], which was previously possible through more established discrete approximative methods, only. The performance of the two-dimensional and axisymmetric version of this method is demonstrated on industrial continuous casting benchmark [5]. Recently, several attempts have been made in order to establish universal way of calculating the laminar and turbulent Navier-Stokes equations by the boundary element method. In [6], a BEM implementation of the well known Patankar-Spalding SIMPLE algorithm is demonstrated, and in [7], a similar, dual reciprocity boundary element method is proposed. Both referenced methods could not be used in the solid-liquid phase change situations due to the assumed constant viscosity and density. This two simplifications allow to use the vector version of the second Green’s theorem for transforming the domain integral of the Laplace equation fundamental solution weighted diffusive term in the momentum equation into boundary integral form. In this work, the requirement to be able to cope strictly temperature dependent viscosity and density, has led to use of the hypersingular formalism which is believed to be the first time applied in connection with the dual reciprocity method and Navier-Stokes equations. The original mass and momentum formulation represented in this paper is for the sake of completeness accompanied with the already established and tested formulation for the energy conservation. Present work could be considered as an upgraded and computationally optimized ”pure” boundary integral version of our classical boundary-domain integral formulations of this problem [8, 9].

2 Energy conservation

Consider a connected fixed domain $\Omega$ with boundary $\Gamma$ occupied by a phase change material described with the temperature dependent density $\varrho_{P}$, specific heat at constant pressure $c_{pP}$ and the thermal conductivity $k_{P}$ of the solid $P = S$ and the liquid $P = L$ phase, and the specific latent heat of the solid-liquid phase change $h_{M}$. The one-phase continuum formulation [10] of the energy transport for the assumed system is

\[
\frac{\partial}{\partial t}(f_{S} \varrho_{S} h_{S} + f_{L} \varrho_{L} h_{L}) + \nabla \cdot (f_{S} \varrho_{S} h_{S} \mathbf{V}_{S} + f_{L} \varrho_{L} h_{L} \mathbf{V}_{L}) = -\nabla \cdot (f_{S} \mathbf{F}_{S} + f_{L} \mathbf{F}_{L}).
\]  

Function $f_{P}$ denotes the temperature dependent volume fraction, $h_{P}$ the specific enthalpy, $\mathbf{V}_{P}$ the velocity, and $\mathbf{F}_{P}$ the heat flux of phase $P$. Since only two phases are present in the system, $f_{S} + f_{L} = 1$. Due to the local thermal equilibrium between the phases, the phase temperatures are equal...
and represented by $T$. The phase change takes place between solidus $T_S$ and liquidus temperature $T_L$. Mean phase change temperature $T_M$ and phase change interval $T_{LS}$ are $T_M = \frac{1}{2} T_S + \frac{1}{2} T_L$, $T_{LS} = T_L - T_S$. Pure substances are modelled by a narrow phase change interval with $T_M$ in this case standing for melting temperature. Constitutive equations for the heat fluxes are based on the Fourier relation $F_p = -k_p \nabla T$, and the two enthalpy-temperature relationships are

$$h_S = \int_{T_{ref}}^{T} c_p S(\theta) d\theta, \quad h_L = h_S(T_S) + \int_{T_S}^{T} c_p L(\theta) d\theta + h_M, \quad (2)$$

with $T_{ref}$ representing the enthalpy reference temperature. The governing equation could be rewritten in the following latent heat source term form

$$\varrho_0 c \frac{\partial T}{\partial t} + \nabla \cdot \Phi = \nabla \cdot (k \nabla T) - \varrho_0 h_{LS} \frac{df_L}{dt}. \quad (3)$$

Specific heat $c$ and thermal conductivity $k$ of the continuum are defined as $\varrho_0 c = f_S h_S \frac{d\varrho_S}{dT} + f_L h_L \frac{d\varrho_L}{dT} + f_S \varrho_S c_p S + f_L \varrho_L c_p L$, $k = f_S k_S + f_L k_L$, the enthalpy difference $h_{LS}$, and the representative density $\varrho_0$ of the system are $\varrho_0 h_{LS} = \varrho_L h_L - \varrho_S h_S$, $\varrho_0 = \frac{1}{2} \varrho_S (T_M) + \frac{1}{2} \varrho_L (T_M)$. The abbreviation in convective term is $\Phi \equiv f_S \varrho_S h_S \nabla S + f_L \varrho_L h_L \nabla L$. We seek the solution of the governing equation for thermal field at final time $t = t_0 + \Delta t$, where $t_0$ represents initial time and $\Delta t$ the positive time increment. The solution is constructed by the initial and boundary conditions that follow. The initial temperature $T(p, t_0)$ at point with position vector $p$ and time $t_0$ is defined through the known function $T_0$; $T(p, t_0) = T_0; p \in \Omega \cup \Gamma$. The boundary $\Gamma$ is divided into not necessarily connected parts $\Gamma^D, \Gamma^N$, and $\Gamma^R$; $\Gamma = \Gamma^D \cup \Gamma^N \cup \Gamma^R$, with Dirichlet, Neumann, and Robin type boundary conditions respectively. These boundary conditions are at point $p$ and time $t_0 \leq t \leq t_0 + \Delta t$ defined through known functions $T_r$, $T_f$, and $H_r$: $T(p, t) = T_r; p \in \Gamma^D$, $\frac{\partial T}{\partial n_r}(p, t) = -\frac{F_r}{k_r}$; $p \in \Gamma^N$, $\frac{\partial T}{\partial n_r}(p, t) = -\frac{H_r}{k_r}(T - T_{r_{ref}})$; $p \in \Gamma^R$, where the heat transfer coefficient $H_r$ and other known functions are allowed to depend arbitrarily on the thermal field. The outward pointing normal on $\Gamma$ is denoted by $n_r(p)$. Equation (3) is rewritten by introducing the Kirchhoff variable

$$T(T) = T_r + \int_{T_r}^{T} \frac{k(\theta)}{k_0} d\theta, \quad (4)$$

defined with representative thermal conductivity $k_0 = \frac{1}{2} k_S (T_M) + \frac{1}{2} k_L (T_M)$, and with the Kirchhoff variable reference temperature $T_r$. Knowledge of the inverse Kirchhoff function $T = T(T)$ is assumed as well. The governing equation is accordingly reformulated as

$$\varrho_0 k_0 \frac{c}{k} \frac{\partial T}{\partial t} + \nabla \cdot \Phi = k_0 \nabla^2 T - \varrho_0 h_{LS} \frac{\partial}{\partial t} f_L, \quad (5)$$
the initial conditions as $T(p, t_0) = T(T_0); p \in \Omega$, and the boundary conditions as $T(p, t) = T(T_1); p \in \Gamma^D$, $\frac{\partial T}{\partial n_T}(p, t) = -\frac{F_T}{k_0}; p \in \Gamma^N$, $\frac{\partial T}{\partial n_T}(p, t) = -\frac{H_T}{k_0}(T(T) - T_{\text{ref}}); p \in \Gamma^R$. The transformed equation (5) with transformed initial and boundary conditions is solved for $\frac{\partial T}{\partial n_T}(p, t_0 + \Delta t); p \in \Gamma^D$, $T(p, t_0 + \Delta t); p \in \Omega \cup \Gamma^N \cup \Gamma^R$, giving the required thermal field through the inverse Kirchhoff function.

### 3 Mass and momentum conservation

The one-phase continuum formulation [10] of the mass and momentum conservation for the assumed system is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0, \quad (6)$$

$$\frac{\partial (\rho \mathbf{V})}{\partial t} + \nabla \cdot (\rho \mathbf{V} \mathbf{V}) = -\nabla P + \nabla \cdot (\mu \text{eff} \nabla \mathbf{V}) + \mathbf{f}, \quad (7)$$

with one-phase density and one-phase velocity defined as

$$\rho = \rho_s \rho_s + \rho_L \rho_L, \quad \rho \mathbf{V} = \rho_s \rho_s \mathbf{V}_s + \rho_L \rho_L \mathbf{V}_L, \quad (8)$$

and $\mu \text{eff}$ and $\mathbf{f}$ denoting the effective viscosity and the body force, respectively. One of the velocities have to be expressed with the other one or otherwise determined in order to close the equation system. Two most simple limiting cases [11] are considered in present paper. The first one represents the disconnected (for example pure equiaxed solidification) solid phase. This limit assumes the solid phase velocity be equal to the liquid phase velocity, i.e., $\mathbf{V}_S = \mathbf{V}_L$. The second limit represents the connected (for example pure columnar solidification) solid phase. This limit assumes the solid velocity be equal to the system velocity, i.e., $\mathbf{V}_S = \mathbf{V}_{\text{sys}}$. The disconnected-solid-limit effective viscosity is modelled through relation

$$\mu \text{eff} = f_s \rho_s \mu_s + f_L \rho_L \mu_L, \quad (9)$$

with $\mu_P$ representing the viscosity of the phase $P$. The adjacent body force is modelled by the Boussinesq approximation with $T_0$ representing the body force reference temperature, $g$ the magnitude of the gravity acceleration

$$\mathbf{f} = -g \rho (1 - \gamma_T(T - T_0)) \mathbf{i}_y, \quad (10)$$

and $\gamma_T$ stands for the coefficient of thermal expansion

$$\gamma_T = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_P. \quad (11)$$
The phase-change process is in this case represented via drastic change of effective viscosity (the solid phase viscosity is set to a large value). The connected-solid-limit effective viscosity is modelled by

$$\mu_{\text{eff}} = \frac{\rho}{\rho_{\text{C}}} \mu_{\text{C}}. \quad (12)$$

The adjacent body force is modelled by the Boussinesq approximation and the mushy zone Darcy porous media assumption

$$f = -g \rho (1 - \gamma_T (T - T_0)) \mathbf{i}_y - \frac{\mu_{\text{C}} \rho}{K} (\mathbf{V} - \mathbf{V}_S). \quad (13)$$

The phase-change process is in this case represented via drastic change of body force, for example through Kozeny-Karman relation

$$K = K_0 \frac{f_{\text{C}}^3}{(1 - f_{\text{C}})^2}, \quad (14)$$

with reference permeability $K_0$. The initial condition for mass and momentum equation represent known velocity field $\mathbf{V}_0$ at time $t_0$

$$\mathbf{V} = \mathbf{V}_0, \quad (15)$$

over the domain $\Omega$, and boundary condition the known velocity field $\mathbf{V}_\Gamma$ at time $t_0 \leq t \leq t_0 + \Delta t$

$$\mathbf{V} = \mathbf{V}_\Gamma, \quad (16)$$

over the boundary $\Gamma$. We seek the solution of the governing equations (6,7) for velocity field at time $t_0 + \Delta t$.

4 Solution procedure - energy conservation

Time discretization of equation (5) is performed with finite differencing

$$\rho_0 k_0 \left( \frac{c_j}{k} \right)^{j+1} \frac{1}{\Delta t} (\mathcal{T}^{j+1} - \mathcal{T}^j) + \nu \nabla \cdot \Phi^{j+1} + (1 - \nu) \nabla \cdot \Phi^j$$

$$= k_0 (\nu \nabla^2 \mathcal{T}^{j+1} + (1 - \nu) \nabla^2 \mathcal{T}^j) - \rho_0 h_{\text{C}}^{j+\nu} \frac{1}{\Delta t} (f_{\text{C}}^{j+1} - f_{\text{C}}^j), \quad (17)$$

where indexes $j$ and $j+1$ represent values at $t_0$ and $t_0 + \Delta t$. Coefficient $\nu = 1$ gives fully implicit, and $\nu = 1/2$ Crank-Nicolson scheme. Since liquid volume fraction, material properties and convective term depend on temperature, iterations over the timestep are inherently required in order to find the solution. These iterations are stopped when the maximum absolute value of the enthalpy difference between two successive iterations in
each of the gridpoints does not exceed some predetermined value \( h_{\text{err}} \). Arbitrary material property \( \varpi \) at iteration level \( m+1 \) and time level \( j + \nu \) is calculated as

\[
m^{j+1} \varpi_{j+1} = \varpi_{j+1}^{m+1} + (1 - \nu) \varpi_j + \nu (T(m^{j+1}T_{j+1})).
\]

The liquid volume fraction at iteration level \( m+1 \) is approximated with the Kirchhoff variable at iteration level \( m+1 \) (Voller-Swaminathan [12] einsatz)

\[
m^{j+1} f_{L_c}^{j+1} = m f_{L_c}^{j+1} + m \left( \frac{k_0}{k} \frac{df_L}{dT} \right)^{j+1} \left( (m^{j+1} - m f_{L_c}^{j+1}) \right).
\]

The convective term at iteration level \( m+1 \) is expanded similarly as

\[
m^{j+1} \Phi_{j+1} = m \Phi_{j+1} + m \left( \frac{k_0}{k} \frac{\partial \Phi}{\partial f_L} \frac{df_L}{dT} \right)^{j+1} \left( (m^{j+1} - m \Phi_{j+1}) \right) + m \left( \frac{\partial \Phi}{\partial \varphi} \frac{d\varphi}{dT} + \frac{\partial \Phi}{\partial \psi} \frac{d\psi}{dT} + \frac{\partial \Phi}{\partial \phi} \frac{d\phi}{dT} \right)^{j+1} \left( m^{j+1} \Phi_{j+1} - m \Phi_{j+1} \right);
\]

\[
\frac{\partial \Phi}{\partial f_L} \frac{df_L}{dT} = (\varphi_L h_L V_L - \varphi_S h_S V_S) \frac{df_L}{dT},
\]

\[
\frac{\partial \Phi}{\partial \varphi} \frac{d\varphi}{dT} = f_p h_p V_p \frac{d\varphi}{dT}, \quad \frac{\partial \Phi}{\partial \phi} \frac{d\phi}{dT} = f_p \varphi_p V_p c_p p.
\]

The final time-discretized form of equation (5) is obtained by inserting the \( m+1 \) iteration level material properties, liquid volume fraction, and convective term expansions in equation (17). The adjacent iterative treatment of boundary conditions is

\[
m^{j+1} T^{j+1} = \nu (m^{j+1} T^{j+1}) + (1 - \nu) T(T^{j+1}); \quad p \in \Gamma^D,
\]

\[
m^{j+1} \frac{\partial T^{j+1}}{\partial n_\Gamma} = -\nu \frac{F_{\Gamma}^{j+1}}{k_0} - (1 - \nu) F_{\Gamma}^{j}; \quad p \in \Gamma^N,
\]

\[
m^{j+1} \frac{\partial T^{j+1}}{\partial n_\Gamma} = -\nu \left[ \frac{m H_{\Gamma}}{m_k} m^{j+1} T - \frac{m H_{\Gamma}}{k_0} (m^{j+1} T_{ref} - m^{j+1} T + \frac{k_0}{m_k} m^{j+1} T) \right]^{j+1}

- (1 - \nu) \left[ \frac{H_{\Gamma}}{k_0} (T - T_{ref}) \right]^{j}; \quad p \in \Gamma^R.
\]

Space discretization is made by weighting the time-discretized equation (5) over domain \( \Omega \) by the fundamental solution of the Laplace equation \( T^* \). The integral type (\( F \) stands for arbitrary scalar and \( G \) for arbitrary vector valued function respectively) \( I_1 \) arises when weighting the transience and source
terms, \( I_2 \) when weighting the convective term, and \( I_3 \) when weighting the diffusive term

\[
I_1 = \int_{\Omega} \mathcal{F}(p) T^*(p; s) d\Omega, \quad I_2 = \int_{\Omega} \nabla \cdot \mathcal{G}(p) T^*(p; s) d\Omega,
\]

\[
I_3 = \int_{\Omega} \nabla^2 \mathcal{F}(p) T^*(p; s) d\Omega. \tag{24}
\]

Let us focus on two-dimensional Cartesian (coordinates \( p_x, p_y \), base vectors \( i_x, i_y \)) geometry and field situations, e.g.,

\[
T^*(p; s) = \frac{1}{2\pi} \log \frac{r_0}{r}, \tag{25}
\]

with \( r^2 = (p_x - s_x)^2 + (p_y - s_y)^2 \). Boundary geometry is approximated by \( N_\Gamma \) straight line segments, and spatial variation of the fields on each of the boundary segments is represented by constant interpolation functions with meshpoints coinciding with the geometrical centres of the straight line segments. Spatial variation of the fields in \( \Omega \) is represented by the \( N_\psi \) global interpolation functions [13] of the form (Einstein summation convention is used in this text, i.e., any index that is repeated twice in a product is summed on, underlined index is not summed on) \( \mathcal{F}(p) \approx \psi_u(p) \xi_u \), collocating \( N = N_\Gamma + N_\Omega \) meshpoints \( p_n \) (first \( N_\Gamma \) collocations points coincide with the boundary meshpoints and the remaining \( N_\Omega \) are arbitrarily distributed in the domain). Coefficients \( \xi_u \) are calculated by constructing [14] a system of \( N_\psi \) algebraic equations \( \Psi \xi = \mathcal{F} \). The vectors are \( \xi = (\xi_1, \xi_2, \ldots, \xi_{N_\psi})^T \) and \( \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{N_\Omega}, 0, \ldots, 0)^T \). The first \( N \) rows of matrix \( \Psi \) are of the form \( (\psi_{u1}, \psi_{u2}, \ldots, \psi_{uN_\psi}) \), and the remaining rows are of the form \( (\psi_{1v}, \psi_{2v}, \ldots, \psi_{Nv}, 0, \ldots, 0) \), where the notation has been shortened to \( \mathcal{F}_n \equiv \mathcal{F}(p_n), \psi_{nu} \equiv \psi_u(p_n) \). Coefficients \( \xi_u \) follow by inverting the system of algebraic equations \( \xi = \Psi^{-1} F \). Selection of simply augmented scaled thin plate splines [15] for \( \psi_u \) is believed to yield an optimal approximating property. The two-dimensional form of these functions is

\[
\psi_n = r_n^2 \log r_n; \quad n=1,2,\ldots,N, \quad \psi_{N+1} = 1, \quad \psi_{N+2} = p_x - p_{0x}, \quad \psi_{N+3} = p_y - p_{0y}. \tag{26}
\]

The integral \( I_3 \) is calculated by using the Green’s second identity

\[
I_{3l} \approx G_{lk} \delta_{ki} \frac{\partial \mathcal{F}_i}{\partial n_l} - H_{lk} \delta_{ki} \mathcal{F}_i - e^*_l \delta_{li} \mathcal{F}_i;
\]

\[
G_{lk} = \int_{\Gamma_k} T^*_l d\Gamma, \quad H_{lk} = \int_{\Gamma_k} \frac{\partial T^*_l}{\partial n_{l\Gamma}} d\Gamma, \tag{27}
\]
where \( k = 1, 2, \cdots N_{\Gamma} \) and \( i, l = 1, 2, \cdots N \). Index \( l \) stands for \( s_l = p_l, \Gamma_k \) represents the \( k \)-th boundary segment, and \( c^*_l \) is equal to \( c^*_l = \frac{1}{2}; s_l \in \Gamma, c^*_l = 1; s_l \in \Omega \). The integrals \( I_1 \) and \( I_2 \) are calculated by defining the harmonic functions \( \nabla^2 \hat{\psi}_u(p) = \hat{\psi}_u(p) \). Consequently, integrals \( I_1 \) and \( I_2 \) could be written in a compact dual reciprocity form

\[
I_{1l} \approx \Psi_{lu} \Psi_{ui}^{-1} \mathcal{F}_l, \quad I_{2l} \approx \Psi_{xi} \Psi_{ni}^{-1} \mathcal{G}_x + \Psi_{yi} \Psi_{ni}^{-1} \mathcal{G}_y;
\]

\[
\Psi_{lu} \approx G_{lk} \delta_{ki} \frac{\partial \hat{\psi}_{iu}}{\partial n_l} - H_{lk} \delta_{ki} \hat{\psi}_{iu} - c^*_l \delta_{li} \hat{\psi}_{iu};
\]

\[
\Psi_{\zeta lu} \approx G_{lk} \delta_{ki} \frac{\partial \hat{\psi}_{iu}}{\partial p_{\zeta}} - H_{lk} \delta_{ki} \hat{\psi}_{iu} - c^*_l \delta_{li} \frac{\partial \hat{\psi}_{iu}}{\partial p_{\zeta}}.
\]  

(28)

Index \( \zeta \) could take values \( x, y \), and indexes \( o, u, \) and \( v \) run over \( o = 1, 2, \cdots N, u, v = 1, 2, \cdots, N^3 \). The formulation of the convective term used in this paper is much more efficient and accurate than the forms used previously [16] in the context of the variable velocity fields. Two-dimensional functions \( \psi_n \) have been found to be

\[
\hat{\psi}_n = \frac{1}{16} r_n^4 \log r_n - \frac{1}{32} r_n^4, n = 1, 2, \cdots, N, \quad \hat{\psi}_{N+1} = \frac{1}{4} (p_x - p_{0x})^2 + \frac{1}{4} (p_y - p_{0y})^2, \]

\[
\hat{\psi}_{N+2} = \frac{1}{6} (p_x - p_{0x})^3, \quad \hat{\psi}_{N+3} = \frac{1}{6} (p_y - p_{0y})^3.
\]  

(29)

After weighting the time discretized governing equation by the fundamental solution of the Laplace equation and calculating the boundary and domain integrals as described, the completely discretized form is structured. Rearrangement of its terms with respect to the boundary conditions gives a system of algebraic equations for solving the unknowns \( m+1 \mathcal{T}_n^{j+1} \) or \( m \frac{\partial \mathcal{T}_n^{j+1}}{\partial n} \) on the boundary, and \( m+1 \mathcal{T}_n^{j+1} \) in the domain. The method does not need any over- or under-relaxation parameters. More details of the derivation and the extensive testing of the present energy conservation solution method are elaborated in [3].

5 Solution procedure - mass and momentum conservation

The convective and diffusive terms are shortened as

\[
\mathbf{C} = \nabla \cdot (\varrho \mathbf{V} \mathbf{V}), \quad \mathbf{D} = \nabla \cdot (\mu_{\text{eff}} \nabla \mathbf{V}).
\]  

(30)

The momentum equation is discretized with respect to time by using finite differencing

\[
\frac{1}{\Delta t} (m \varrho^j \mathbf{V}^{j+1} - (\varrho \mathbf{V})^j) = -\nabla m \mathbf{P}^{j+t} + m \mathbf{S}^{j+t},
\]  

(31)
with

\[ S = -C + D + f, \quad S^{j+1} = \iota S^{j+1} + (1 - \iota) S^j. \]  

The velocity field at time level \( j + 1 \) could be calculated iteratively (iteration level \( m + 1 \)) from equation (31) in case the pressure gradient and term \( S \) are known at iteration level \( m \). The iterations are stopped when the maximum absolute value of the velocity components difference between two successive iterations in each of the gridpoints does not exceed some predetermined value \( V_{\text{err}} \). An under-relaxation with factor \( \lambda \) might be required to achieve the convergence

\[ m+1 \mathbf{V}^{j+1} = m \mathbf{V}^{j+1} + \lambda (m+1 \mathbf{V}^{j+1} - m \mathbf{V}^{j+1}). \]  

The calculation of term \( S \) requires the following global interpolation functions manipulations. The components of the convective term in two-dimensional Cartesian frame are (left compressible and right incompressible media)

\[ C_\zeta = \frac{\partial}{\partial p_x} (\varrho V_x V_\zeta) + \frac{\partial}{\partial p_y} (\varrho V_y V_\zeta), \quad C_\zeta = \varrho V_x \frac{\partial}{\partial p_x} V_\zeta + \varrho V_y \frac{\partial}{\partial p_y} V_\zeta. \]  

The components of the diffusive term are

\[ D_\zeta = \frac{\partial}{\partial p_x} \mu_{\text{eff}} \frac{\partial}{\partial p_x} V_\zeta + \mu_{\text{eff}} \frac{\partial^2}{\partial p_x^2} V_\zeta + \frac{\partial}{\partial p_y} \mu_{\text{eff}} \frac{\partial}{\partial p_y} V_\zeta + \mu_{\text{eff}} \frac{\partial^2}{\partial p_y^2} V_\zeta. \]  

The components of the convective term in point \( p_i \) could be simply calculated through the global interpolation functions as

\[ C_{\zeta i} = \frac{\partial}{\partial p_x} \psi_{i n} \Psi_{n j}^{-1} \varrho_j V_{x i} V_{\zeta j} + \frac{\partial}{\partial p_y} \psi_{i n} \Psi_{n j}^{-1} \varrho_j V_{y i} V_{\zeta j}, \]  

\[ C_{\zeta i} = \varrho_i V_{x i} \frac{\partial}{\partial p_x} \psi_{i n} \Psi_{n j}^{-1} V_{\zeta j} + \varrho_i V_{y i} \frac{\partial}{\partial p_y} \psi_{i n} \Psi_{n j}^{-1} V_{\zeta j}, \]  

where the equation (36) stands for compressible and equation (37) for incompressible case, respectively. Similarly, the components of the diffusive term could be calculated as

\[ D_{\zeta i} = (\frac{\partial}{\partial p_x} \psi_{i n} \Psi_{n j}^{-1} \mu_{\text{eff} j}) (\frac{\partial}{\partial p_x} \psi_{i n} \Psi_{n j}^{-1} V_{\zeta j}) + \mu_{\text{eff} j} \frac{\partial^2}{\partial p_x^2} \psi_{i n} \Psi_{n j}^{-1} V_{\zeta j} + \frac{\partial^2}{\partial p_y^2} \psi_{i n} \Psi_{n j}^{-1} V_{\zeta j}, \]  

\[ + (\frac{\partial}{\partial p_y} \psi_{i n} \Psi_{n j}^{-1} \mu_{\text{eff} j}) (\frac{\partial}{\partial p_y} \psi_{i n} \Psi_{n j}^{-1} V_{\zeta j}) + \mu_{\text{eff} j} \frac{\partial^2}{\partial p_y^2} \psi_{i n} \Psi_{n j}^{-1} V_{\zeta j}. \]
The calculation of body force \( f(p) = f_i \) does not involve global interpolation functions (no spatial derivatives!). The calculation of the pressure gradient requires the following manipulations. The divergence of the momentum equation gives [17] the consistent pressure Poisson equation

\[
\nabla^2 P = S + \nabla \cdot S,
\]

where the first term on the right hand side arises from use of the mass conservation equation

\[
S = -\nabla \frac{\partial}{\partial t} (\rho \mathbf{V}) = -\frac{\partial}{\partial t} \nabla \cdot (\rho \mathbf{V}) = \frac{\partial^2}{\partial t^2} \rho,
\]

and equals 0 in case of incompressible media. It is (when needed) calculated on a three time-level basis

\[
S^{j+i} = \frac{\Delta t^{j-1} \rho^{j+1} - (\Delta t^{j-1} + \Delta t^j) \rho^j + \Delta t^j \rho^{j-1}}{(\Delta t^j)^2 \Delta t^{j-1}}.
\]  

The pressure Poisson equation is solved by using weighting with the fundamental solution of the Laplace equation over the domain \( \Omega \)

\[
\int_{\Gamma} \frac{\partial P}{\partial n_{\Gamma}} T^* d\Gamma - \int_{\Gamma} P \frac{\partial T^*}{\partial n_{\Gamma}} d\Gamma - c^* P(s) = \int_{\Omega} ST^* d\Omega - \int_{\Omega} \nabla \cdot S T^* d\Omega.
\]  

The left side of equation (42) is discretized in sense of integral type \( I_3 \), the first integral on the right hand side in sense of integral type \( I_1 \), and the second integral on the right hand side in sense of the integral type \( I_1 \). Neumann pressure boundary conditions over the boundary \( \Gamma \) are determined simply as

\[
m \frac{\partial P}{\partial n_{\Gamma}} = -\frac{1}{\Delta t} (m (\rho \mathbf{V})^{j+1} - (\rho \mathbf{V})^j) \cdot \mathbf{n}_{\Gamma} + m S^{j+i} \cdot \mathbf{n}_{\Gamma},
\]

and the Dirichlet pressure boundary condition \( P = P_0 \), with \( P_0 \) representing the known reference pressure, have to be assumed at least one arbitrary boundary point, \( \mathbf{p} = \mathbf{p}_0 \). Discretized equation (42) is used to solve the complete pressure and pressure gradient distribution on the boundary. Pressure gradient in the domain \( \Omega \) is determined by taking the gradient of the equation (42) acting on the source coordinate system. The following integral equation for explicit calculation of the pressure gradient in domain points is valid

\[
\int_{\Gamma} \frac{\partial P}{\partial n_{\Gamma}} \nabla T^* d\Gamma - \int_{\Gamma} P \nabla \frac{\partial T^*}{\partial n_{\Gamma}} d\Gamma + \nabla P(s) = \int_{\Omega} S \nabla T^* d\Omega - \int_{\Omega} \nabla \cdot S \nabla T^* d\Omega.
\]
The discretized form of upper equation leads to the following integral types
\[ J_1 = \int_\Omega \mathcal{F}(p) \nabla T^*(p; s) \, d\Omega, \quad J_2 = \int_\Omega \nabla \cdot \mathcal{G}(p) \nabla T^*(p; s) \, d\Omega, \]
\[ J_3 = \int_\Omega \nabla^2 \mathcal{F}(p) \nabla T^*(p; s) \, d\Omega. \]  
(45)

Integral \( J_3 \) could be calculated as
\[ J_{3\zeta l} \approx -G_{\zeta lk} \delta_{ki} \frac{\partial F_i}{\partial n_\Gamma} + H_{\zeta lk} \delta_{ki} F_i - c^*_l \delta_{li} \frac{\partial F_i}{\partial p_\zeta}; \]
\[ G_{\zeta lk} = \int_{\Gamma_k} \frac{\partial T^*_l}{\partial p_\zeta} \, d\Gamma, \quad H_{\zeta lk} = \int_{\Gamma_k} \frac{\partial T^*_l}{\partial p_\zeta} \frac{\partial n_\Gamma}{\partial n_\Gamma} \, d\Gamma, \]  
(46)

and integrals \( J_1 \) and \( J_2 \) could be written in a compact dual reciprocity form (valid exclusively for the simple boundary geometry and fields discretization in this paper)
\[ J_{1\zeta l} \approx \Psi_{\zeta lu} \Psi_{ui}^{-1} F_i, \quad J_{2\zeta l} \approx \Psi_{x\zeta ln} \Psi_{ni}^{-1} G_{ix} + \Psi_{y\zeta ln} \Psi_{ni}^{-1} G_{iy}; \]
\[ \Psi_{\zeta lu} \approx -G_{\zeta lk} \delta_{ki} \frac{\partial \hat{\psi}_{iu}}{\partial n_\Gamma} + H_{\zeta lk} \delta_{ki} \hat{\psi}_{iu} - c^*_l \delta_{li} \frac{\partial \hat{\psi}_{iu}}{\partial p_\zeta}, \]
\[ \Psi_{x\zeta lu} \approx -G_{\zeta lk} \delta_{ki} \frac{\partial \hat{\psi}_{iu}}{\partial p_x} + H_{\zeta lk} \delta_{ki} \frac{\partial \hat{\psi}_{iu}}{\partial p_x} - c^*_l \delta_{li} \frac{\partial^2 \hat{\psi}_{iu}}{\partial p_\zeta \partial p_x}, \]
\[ \Psi_{y\zeta lu} \approx -G_{\zeta lk} \delta_{ki} \frac{\partial \hat{\psi}_{iu}}{\partial p_y} + H_{\zeta lk} \delta_{ki} \frac{\partial \hat{\psi}_{iu}}{\partial p_y} - c^*_l \delta_{li} \frac{\partial^2 \hat{\psi}_{iu}}{\partial p_\zeta \partial p_y}. \]  
(47)

The momentum and energy equations are coupled through the body force in the momentum equation and the convective term in the energy equation. Consequently, the two solution procedures, described in previous two paragraphs, have to be interchangeably applied during timestep.

### 6 Conclusions

The formulation of the dual reciprocity boundary element method for coping with the extremely non-linear coupled energy, mass, and momentum transport in solid-liquid phase change systems is developed. Such type of complex physical models could be previously solved by more established numerical methods only. The governing equations are of the one-phase form.
and the considered structure has been assumed general enough to allow
the implementation of the closure relations for different systems with solid
and liquid phase in addition to the two explicitly tackled limiting cases.
The primitive variables have been retained in all of the discretization equa-
tions. The energy part of the method has been already successfully tested
in detail. The mass and momentum part of the proposed method are now
subject to numerical implementation and extensive step-by-step checking.
The Darcy porous media natural convection sub-problem has been already
accomplished [18].

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