Modeling of the movement of electrophoretic cake surface
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Abstract
A finite difference procedure with moving nodal points is presented to analyze a one-dimensional accreting compressible clay cake because of electrical and gravitational forces. The procedure involves solution of a second order, non-linear partial differential equation for cake compression where the cake thickness is modified by employing a no-jump condition at the suspension/clay cake interface. Analytical solutions are obtained for the final cake thickness and steady state porosity distribution. The results of the model are presented graphically and the numerical procedure is verified with the exact solutions.

1 Introduction
Leaks in geomembrane liners underneath the waste impoundments cause subsurface contamination. Darilek [3] proposed an electrophoretic sealing method, which is a safe and cost-effective repair technique, to seal leaks in in-service liners. Electrophoresis is an electrokinetic phenomenon that involves migration of electrically charged suspended particles under the influence of an externally applied electric field. When a direct current voltage is impressed across the electrically insulated geomembrane liner by placing an electrode (cathode) in the liquid and another electrode (anode) in the earth outside the impoundment, current flows through the leaks in the plastic liner establishing a strong electric field at the leaks. When a suspension prepared from clay, e.g. bentonite, is introduced into the impoundment, the negatively charged clay
particles migrate toward the anode under the influence of the applied electric field. These particles deposit at the leak of the geomembrane liner forming a clay cake. The electric and gravitational forces further consolidate this compressible cake forming a relatively impermeable barrier to seal the leaks.

In this paper we simulate one-dimensional finite strain compressible cake formation that results from electrical and gravitational forces using the governing equation derived by Kambham et al. [4]. The governing equation for cake compression, which is a second-order, non-linear partial differential equation, is solved using a finite difference technique with moving nodal points. The solid fluxes on either side of suspension/clay cake interface are coupled using a no-jump condition. Analytical solutions are obtained for steady state condition. These solutions, which can be used to determine the final cake thickness and steady state distribution of volume fraction of solids, are compared with the simulation results of the moving boundary problem. The numerical results are in good agreement with the exact solutions.

2 Electrokinetic Compression of Clay Cake

In our formulation we use the volume fraction of solids \( \phi \) as the parameter to represent particle concentration. Fluid and solid phases are assumed to be incompressible, whereas the solid matrix as a whole is compressible because of changes in the volume fraction of solids which is defined as the ratio of the volume of solids to the total volume. It is assumed that the domain where volume fraction of solids is greater than a critical value \( \phi_c \), which is known as the volume fraction of solids at zero effective stress or "stress-free state" [2], forms clay cake. The governing equation presented in this section, which describes compression of cake domain, is used to simulate the accreting cake after taking into account deposition of solids from suspension zone.

The one-dimensional mass balance equation for solid particles in a deformable saturated porous medium is expressed as

\[
-\frac{\partial}{\partial x}(\phi v_s) - \frac{\partial \phi}{\partial t}
\]

where \( v_s \) is the velocity of solid particles, \( x \) is measured positive upward and \( t \) is time. The velocity of solid particles for electrokinetic consolidation of a saturated porous medium on a fixed impervious base is given by [4]

\[
v_s = -\frac{k}{\mu} \left[ \frac{1}{\alpha \phi} \frac{\partial \phi}{\partial x} + (\gamma_s - \gamma_w) \phi \right] + U \cdot V' \phi \left( \frac{1 - \phi}{\phi} \right)
\]

In equation (2), \( k \) is the intrinsic permeability \( (m^2) \), \( \mu \) is the viscosity of water...
Free and Moving Boundary Problems

(kN-s/m²), \(a\) is the coefficient of matrix compressibility (m²/kN), \(\gamma_s\) is the unit weight of solid particles (kN/m³), \(\gamma_w\) is the unit weight of water (kN/m³), \(U'\) is the mobility of water in suspension (m²V⁻¹s⁻¹) and \(V\) is the voltage gradient that results from externally applied electric field (V/m). Substitution of equation (2) in (1) yields

\[
\frac{\partial}{\partial x} \left[ k \left( \frac{1}{\alpha} \frac{\partial \phi}{\partial x} + (\gamma_s - \gamma_w) \phi^2 \right) - U' V' \phi_0 (1 - \phi) \right] - \frac{\partial \phi}{\partial t} = 0 \quad 0 \leq x \leq L(t) \tag{3}
\]

where \(L(t)\), the thickness of clay cake at time \(t\), represents the moving boundary at the top because of cake compression only. Equation (3) is the governing equation for one-dimensional finite strain electrokinetic consolidation of a clay cake caused by electrical and gravitational forces.

For an initial volume fraction of solids distribution \(\phi_i\), and an initial cake thickness \(L_i\), respective initial conditions are

\[
\phi(x,0) - \phi_i(x) \quad L(t)|_{x=0} = L_i \tag{4}
\]

Since velocity of solids at the base is zero, form equation (2) we write the lower boundary condition as

\[
\left[ k \left( \frac{1}{\alpha} \frac{\partial \phi}{\partial x} + (\gamma_s - \gamma_w) \phi^2 \right) - U' V' \phi_0 (1 - \phi) \right]_{x=0} = 0 \tag{5}
\]

As solids at the top are at stress-free state, the upper boundary condition is

\[
[\phi(x,t)]_{x=L(t)} = \phi_0 \tag{6}
\]

The velocity of the moving boundary because of cake compression is the same as that of a solid particle on the moving boundary. Then, from equation (2) we can write

\[
\frac{dL}{dt} = \left[ -k \left( \frac{1}{\alpha} \frac{\partial \phi}{\partial x} + (\gamma_s - \gamma_w) \phi \right) + U' V' \phi_0 \left( \frac{1 - \phi}{\phi} \right) \right]_{x=L(t)} \tag{7}
\]

Equation (3) with initial and boundary conditions (4-7) describes a moving boundary problem for finite strain electrokinetic consolidation.

3 Numerical Procedure for Cake Compression

Assuming constant material properties, equation (3) is linearized as

\[
A \frac{\partial^2 \phi}{\partial x^2} + (2B \phi_i - C) \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial t} \tag{8}
\]
where

$$A = \frac{k}{\mu \alpha}, \quad B = (\gamma_s - \gamma_w) \frac{k}{\mu}, \quad C = -U' V' \phi_o$$

and $\phi_i$ is the linearized value of $\phi$ which is iterated until the solution converges. Similarly, the linearized form of the lower boundary condition (equation (5)) is

$$\left[ A \frac{\partial \phi}{\partial x} + (B \phi_i - C) \phi + C \right]_{x=0} = 0 \quad (9)$$

Equation (8) along with the initial and boundary conditions given by (4), (6) and (9) are solved using implicit finite difference scheme. The discretization involves central difference in space and forward difference in time. Since the total mass of the solids in the cake domain is constant, the solution must satisfy

$$\int_0^{U(t+\Delta t)} \phi(x,t+\Delta t) \, dx - \int_0^{U(t)} \phi(x,t) \, dx = 0 \quad (10)$$

Note that equation (10) does not include formation of fresh sediment at the top. The fresh sediment formation at the cake surface is discussed in section 4. Equation (10) is discretized as

$$\int_{X_i(t+\Delta t)}^{X_{i+1}(t+\Delta t)} \phi(x,t+\Delta t) \, dx - \int_{X_i(t)}^{X_{i+1}(t)} \phi(x,t) \, dx = 0 \quad (11)$$

where $i$ represents the node number and $X_i$ is the distance from the base. Assuming a linear variation in $\phi$ between any two adjacent nodes, we find the modified spacings between nodes $i$ and $i+1$ from

$$\Delta x_i(t+\Delta t) = \frac{\phi_{i+1}(t+\Delta t) + \phi_i(t)}{\phi_{i+1}(t+\Delta t) + \phi_i(t+\Delta t)} \Delta x_i(t) \quad (12)$$

where $\Delta x_i$ represents grid spacing between nodes $i$ and $i+1$. From equation (12), the new position of a nodal point $i$ from the base is

$$X_i(t+\Delta t) = \sum_{j=0}^{i-1} \Delta x_j(t+\Delta t) \quad 0 < i \leq N+1 \quad (13)$$

where $N$ is the number of grid spacings in the cake. In equation (13), $i=N+1$ corresponds to the thickness of clay cake at time $t+\Delta t$, i.e., $L(t+\Delta t)$. 
It should be noted that the numerical procedure presented in this section describes the movement of cake surface caused by cake compression only. The build-up of fresh sediment at the top is discussed in the following sections.

4 Fresh Cake Formation

The movement of cake surface has two components. The first one is caused by the compression of cake itself which tends to decrease the cake thickness; whereas, the other component results from the formation of fresh sediment which tends to increase the cake thickness. In most of the cases, the second component dominates and the cake thickness increases as long as there is sedimentation. Assuming the velocity of particles in the suspension side is much greater compared with the velocity of solids on the cake surface because of compression, after employing a no-jump condition, the thickness of fresh cake formed in time increment $\Delta t$ is written as [4]

$$\delta L_{\text{sus}} = \frac{\phi_{\text{sus}} v_{\text{sus}}}{\phi_{\text{sus}} - \phi_o} \Delta t$$  \hspace{1cm} (14)

where the subscript sus refers to the suspension zone. For a monodispersed suspension with a uniform initial particle concentration, $v_{\text{sus}}$ and $\phi_{\text{sus}}$ are constants.

The velocity of a discrete particle in the suspension ($u_0$) is taken as the algebraic sum of the electrophoretic velocity ($u_{ep}$) and Stoke's settling velocity ($u_s$). When a voltage gradient of $V$ is applied, electrophoretic velocity $u_{ep} = mV$ where $m$ is the electrophoretic mobility of a particle ($m$ is the velocity of particle per unit voltage gradient). We employ the empirical relationship given by Richardson & Zaki [1] to account for the effect of particle concentration on the electrophoretic settling velocity of particles in the suspension zone. Then, the velocity of particles in suspension is given by

$$v_{\text{sus}} = u_0 (1 - \phi_{\text{sus}})^{4.65}$$  \hspace{1cm} (15)

5 Simulation of Cake Formation and Compression

Simulation runs are performed for a monodispersed suspension with uniform initial concentration. The initial thickness of the clay cake is taken as zero. First we calculate the thickness of cake $\delta L_{\text{sus}}$ from equation (14) and assume a uniform volume solid fraction distribution of $\phi_0$ in this cake domain. Then the clay cake is divided into $M$ segments, and the nodal values of volume solid
fraction in the cake are calculated using the numerical procedure presented in section 3. The grid spacings are modified from equation (12), and the new compressed thickness is computed using equation (13). In the next time step the cake thickness increases by $\delta L_{\text{sus}}$ because of the formation of fresh sediment. We put an additional nodal point at location $X_M(t+\Delta t) + \delta L_{\text{sus}}$ and assign $\phi_{M+1} = \phi_o$. Thus, during sedimentation, after each time step the number of nodes increases by one. This procedure is repeated until no solid particles are left in the suspension zone. Once all the solids become part of the clay cake, the cake undergoes electrokinetic compression only until the steady state condition is reached.

6 Analytical Solutions for Steady State Condition

Substituting $v_s = 0$ in equation (2) gives the steady state equation for electrokinetic consolidation as

$$A \frac{d\phi}{dx} + B\phi^2 + C(1-\phi) = 0 \quad 0 \leq x \leq L_\infty$$  \hspace{1cm} (16)

where $L_\infty$ is the final thickness and the constants $A$, $B$ and $C$ as are defined in section 3. It should be noted that the suspension that forms clay cake at the bottom eventually reaches the steady state condition given by equation (16). When $4B > C$, integration of equation (16) with the boundary condition $\phi = \phi_o$ at $x = L_\infty$ gives the steady state distribution of volume solid fraction as

$$\phi = \frac{\Delta}{B} \tan \left[ \frac{\Delta}{A}(L_\infty - x) \right] + \phi_o \quad 0 \leq x \leq L_\infty$$  \hspace{1cm} (17)

where $\Delta^2 = (4BC-C^2)/4$. The final thickness is obtained by conserving the total mass of solids in the domain as

$$\int_0^{L_\infty} \phi \, dx = \phi_{\text{sus}} H$$  \hspace{1cm} (18)

where $H$ is the initial height of suspension. Substitution of equation (17) in (18) yields an implicit expression for $L_\infty$ as

$$\cos \left( \frac{\Delta}{A} L_\infty \right) = \exp \left[ - \frac{B}{A} \left( \phi_{\text{sus}} H - \phi_o L_\infty \right) \right]$$  \hspace{1cm} (19)

Equations (19) and (17) form the steady state solutions for the final electrophoretic cake thickness and volume solid fraction distribution, respectively.
7 Results and Discussion

The following parameters are used to form a compressible cake: $\phi_{\text{ss}} = 0.0050$, $\phi_o = 0.0625$, $H = 1.0$ m, $V' = -500$ V/m, $u_o = 3.648E-04$ m/s, $\gamma_s = 26.487$ kN/m$^3$, $\gamma_w = 9.810$ kN/m$^3$, $\mu = 1.0E-06$ kN-s/m$^2$, $\alpha = 1.0$ m$^2$/kN, $k = 1.0E-12$ m$^2$ and $U* = 3.0E-08$ m$^2$V$^{-1}$s$^{-1}$. The simulation results are shown in Figures 1 & 2. Figure 1 illustrates the cake build-up with time and the final thickness of the cake is 0.0558 m. On the other hand, the value of $L_*$ solved from equation (19) is equal to 0.0563 m. Note that an incompressible cake with a uniform volume solid fraction of $\phi_o$ forms a 0.08-m-thick clay cake. Figure 2 shows the volume solid fraction profiles at five different times and the results of steady state analytical solution (equation (17)). A comparison of the steady state profiles show that the simulation results are in good agreement with the exact solution.

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References

Figure 1: Temporal variation of cake thickness.

Figure 2: Profiles of volume fraction of solids.