Existence of solution and numerical simulation for an elastic journal bearing
J. Durany, G. García, C. Vázquez
Department of Applied Mathematics, University of Vigo, 36280 Vigo, Spain

ABSTRACT

A particular elastohydrodynamic lubrication problem is modelled in this work. The presence of elasticity, lubrication and cavitation gives place to a non linear coupled system of variational equations. An existence result is concluded by means of a constructive algorithm that decouples the elastic hinged plate biharmonic equation and the lubrication-cavitation Elrod-Adams free boundary problem.

INTRODUCTION

Since the presentation of the governing equations for the lubrication problem in the hydrodynamic case (see Reynolds [13]) a lot of effort has been done in the domain of elastohydrodynamic lubrication in three areas: mathematical modelling, theoretical analysis and numerical solution algorithms. In this work the authors try to contribute in the previous three aspects for a particular elastohydrodynamic problem concerning to journal bearing devices.

Modellization requires to take into account the following main features of the physical problem: the fluid hydrodynamic displacement, the formation of bubbles inside the fluid (cavitation) and the deformation of the boundary surfaces. The first physical aspect is mathematically modelled by Reynolds’ equation formally justified from Stokes’ equations by Bayada-Chambat [2]. The second aspect arises when the values of the pressure of the lubricant are below the one of saturation pressure. In this case the presence of air bubbles inside the lubricant makes Reynolds’ equation no longer valid in this part of the domain (cavitation region). Between the different cavitation models Elrod-Adams’ one has lately revealed the more realistic (see Bayada-Chambat [3]). The third phenomenon is the elastic deformation of the surfaces in contact by the effect of the fluid pressure. The particular characteristics of the contact and the surfaces justify the use of different elasticity equations. In elastohydrodynamic lubrication contacts the three aspects are coupled each other. Therefore the resulting equations that govern each problem are also coupled.

In this paper the lubrication problem that appears in a journal bearing device with an elastic thin bearing is considered (see Figure 1.a.). The thinness of the bearing allows to approximate the elastic behaviour of it by means of plates equation. A similar problem has been mathematically studied in Cimatti
by using a variational inequality model for the cavitation. Herein, we consider the Elrod-Adams model for cavitation and the resulting coupled problem is posed. The existence of solution result is obtained by means of an algorithm that essentially decouples the hydrodynamic and the elastic part of the problem.

The numerical algorithm here proposed follows the theoretical proof of existence and it requires the solution of a nonlinear elliptic second order free boundary problem and a linear fourth order plate equation. For the first one we follow the idea already developed in Bermúdez-Durany [5] and Durany-Vázquez [9] in the hydrodynamic case (i.e. rigid surfaces). For the biharmonic equation with Dirichlet and periodic boundary conditions a mixed formulation that involves the solution of second order elliptic problems is implemented (see Glowinski-Pironneau [11]).

![Diagram of a journal bearing device](image)

Fig. 1.: a) Elastic journal bearing with thin bearing. b) Bidimensional domain of the problem.

**Formulation of the Equations.**

The journal bearing device basically consists in a cylindrical shaft that rotates inside a cylindrical bearing. The gap between them is lubricated by means of an isoviscous and incompressible fluid supplied through a circumferential groove (see Figure 1.a). The consideration of a thin bearing and the use of Elrod-Adams model for cavitation leads to the following set of equations

Find \((p, \theta, w)\) such that:

\[
\frac{\partial}{\partial x} ((h + w) \frac{\partial p}{\partial x}) + \frac{\partial}{\partial y} ((h + w) \frac{\partial p}{\partial y}) = 6\nu_0 s \frac{\partial}{\partial x} (h + w) \quad p > 0 \quad \text{and} \quad \theta = 1 \quad \text{in} \quad \Omega^+ 
\]

\[
\frac{\partial}{\partial x} (\theta (h + w)) = 0, \quad p = 0 \quad \text{and} \quad 0 \leq \theta \leq 1 \quad \text{in} \quad \Omega_0 
\]

\[
(h + w)^3 \frac{\partial p}{\partial n} = 6\nu_0 (1 - \theta) (h + w) \cos(n,i), \quad p = 0 \quad \text{on} \quad \Sigma 
\]

\[
p = 0 \quad \text{on} \quad \Gamma 
\]
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\( p = p_a \) on \( \Gamma_a \)  \hfill (5)

\( p \) is \( 2\pi - x \) periodic  \hfill (6)

\( \eta \Delta^2 w = p \) in \( \Omega \)  \hfill (7)

\( w = \Delta w = 0 \) on \( \Gamma \cup \Gamma_a \)  \hfill (8)

\( w \) and \( \Delta w \) \( 2\pi - x \) periodic  \hfill (9)

where the unknowns \( p, \theta \) and \( w \) denote the fluid pressure, the saturation function and the deformation of the elastic bearing respectively.

The data are the initial rigid gap \( h \), the constant angular velocity \( s \), the viscosity \( \nu_0 \) and the supply pressure \( p_a \) and the flexure rigidity \( \eta \) of the bearing. The rigid gap is classically approximated in terms of the difference between the radii \( C \) and the eccentricity \( \beta \) (\( 0 < \beta < 1 \)) in the form

\[ h(x) = C(1 + \beta \cos(x)) \]  \hfill (10)

The sets that appear in the strong formulation of the problem are, see Figure 1.b.,

\[ \Omega = (0, 2\pi) \times (0, 1) \]

\[ \Omega_0 = \{(x, y) \mid p(x, y) = 0\} \]

\[ \Gamma = \{(x, y) \in \partial \Omega / y = 0\} \]

\[ \Gamma_a = \{(x, y) \in \partial \Omega / y = 1\} \]

\[ \Gamma_{per} = \{(x, y) \in \partial \Omega / x = 0 \text{ or } x = 2\pi\} \]

where \( \Omega \) corresponds to the mean plane of the gap (so \( x \) represents the radial coordinate and \( y \) is the axial coordinate). Equations (1)-(6) correspond to the mathematical model of a thin film fluid displacement considering the possibility of cavitation. Equations (7)-(9) govern the hinged plate behaviour of the thin bearing. The introduction of the new variable \( p^* \) defined by

\[ p^*(x, y) = p(x, y) - p_a y \]  \hfill (11)

provides a suitable weak formulation of Equations (1)-(9) given by

\[ \text{Find } (p^*, \theta, w) \in M_0 \times L^\infty(\Omega) \times L_0 \text{ such that :} \]

\[ \int_{\Omega} (h + w)^3 \nabla p^* \nabla \varphi \, dx \, dy = \int_{\Omega} (h + w) \theta \frac{\partial \varphi}{\partial x} \, dx \, dy \]

\[ - p_a \int_{\Omega} (h + w)^3 \frac{\partial \varphi}{\partial y} \, dx \, dy \quad \forall \varphi \in M_0 \]  \hfill (12)

\[ \eta \int_{\Omega} \Delta w \Delta \psi \, dx \, dy = \int_{\Omega} (p^* + p_a y) \psi \, dx \, dy \quad \forall \psi \in L_0 \]  \hfill (13)
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\[ p^* \geq -p_a y \quad (14) \]

\[ H(p^* + p_a y) \leq \theta \leq 1 \quad (15) \]

where \( H \) denotes the Heaviside function and

\[ M_0 = \{ \varphi \in H^1(\Omega) / \varphi = 0 \text{ on } \Gamma \cup \Gamma_a \text{ and } \varphi \text{ } 2\pi - x \text{ periodic} \} \quad (16) \]

\[ L_0 = \{ \psi \in H^2(\Omega) / \psi = \Delta \psi = 0 \text{ on } \Gamma \cup \Gamma_a \text{ and } \psi \text{ } 2\pi - x \text{ periodic} \} \quad (17) \]

where \( H^1(\Omega) \) and \( H^2(\Omega) \) denote the classical Sobolev spaces.

EXISTENCE OF SOLUTION.

The mathematical analysis consists in the proof of existence of solution for Equations (12)-(15). It is based on the construction of an algorithm that generates a sequence \( \{(p_n, \theta_n, w_n)\} \) in the space \( M_0 \times L^\infty(\Omega) \times L_0 \) converging to the solution.

Next paragraph is devoted to the description of the algorithm:

STEP 1 : Let be \( p_0^* = w_0 = 0 \) and \( \theta_0 = 1 \)

**Problem 1.1** - Find \( (p_1^*, \theta_1) \in M_0 \times L^\infty(\Omega) \) such that

\[
\int_\Omega (h + w_0)^3 \nabla p_1^* \cdot \nabla \varphi \, dx \, dy = \int_\Omega (h + w_0) \theta_1 \frac{\partial \varphi}{\partial x} \, dx \, dy - p_a \int_\Omega (h + w_0)^3 \frac{\partial \varphi}{\partial y} \, dx \, dy \quad \forall \varphi \in M_0 \quad (18)
\]

\[ p_1^* \geq -p_a y \quad (19) \]

\[ H(p_1^* + p_a y) \leq \theta_1 \leq 1 \quad (20) \]

**Problem 1.2** - Find \( w_1 \in L_0 \) such that

\[
\eta \int_\Omega \Delta w_1 \Delta \psi \, dx \, dy = \int_\Omega (p_1^* + p_a y) \psi \, dx \, dy \quad \forall \psi \in L_0 \quad (21)
\]

This first step takes into account that \( w_0 = 0 \) and that \( h(x) \) is a \( 2\pi - x \) periodic function. So

\[
\int_\Omega (h + w_0)^3 \frac{\partial \varphi}{\partial y} \, dx \, dy = 0 \quad (22)
\]
\[ 0 < C(1 - \beta) \leq h \leq C(1 + \beta) \quad (23) \]

The previous arguments reduce Problem 1.1 to the one already treated in Alvarez [1]. Existence, uniqueness of solution and the regularity property \( p_1^* \in C(\Omega) \) have been proved. The classical results of existence and uniqueness of solution for Problem 1.2 can be found in Rektoris [12], for example. In Cimatti [8] the regularity property \( w_1 \in H^3(\Omega) \) is obtained. The fact that \( p_1 \) is non negative and the application of the weak maximum principle twice conclude that \( w_1 \) is also nonnegative (provided the boundary conditions imposed for \( w_1 \)).

The consideration of \( p_1^* \) as test function in Problem 1.1, the application of the Holder inequality and the lower bound of \( h \) lead to the \( H^1(\Omega) \) estimate for \( p_1^* \)

\[
\left[ \int_{\Omega} h^2 | \nabla p_1^* |^2 \, dx \, dy \right]^{\frac{1}{2}} \leq K(\Omega, \beta, C) \quad (24)
\]

where \( K(\Omega, \beta, C) \) is a constant. The above results and the equation (21) conclude that (see Bayada-Durany-Vázquez [4] for details)

\[
\| p_1 \|_{H^1(\Omega)} \leq K_1(\Omega, \beta, C) \quad (25)
\]

\[
\| w_1 \|_{H^2(\Omega)} \leq K_2(\Omega, \beta, C) \quad (26)
\]

STEP n :

**Problem n.1** - Find \((p_n^*, \theta_n) \in M_0 \times L^\infty(\Omega)\) such that

\[
\int_{\Omega} (h + w_{n-1})^3 \nabla p_n^* \cdot \nabla \phi \, dx \, dy = \int_{\Omega} (h + w_{n-1}) \theta_n \frac{\partial \phi}{\partial x} \, dx \, dy - p_a \int_{\Omega} (h + w_{n-1})^3 \frac{\partial \phi}{\partial y} \, dx \, dy \quad \forall \phi \in M_0 \quad (27)
\]

\[
p_n^* \geq -p_a y \quad (28)
\]

\[
H(p_n^* + p_a y) \leq \theta_n \leq 1 \quad (29)
\]

**Problem n.2** - Find \( w_n \in L_0 \) such that

\[
\eta \int_{\Omega} \Delta w_n \Delta \psi \, dx \, dy = \int_{\Omega} (p_n^* + p_a y) \psi \, dx \, dy \quad \forall \psi \in L_0 \quad (30)
\]

At the end of the \((n - 1)\)-th step the corresponding \( n \)-independent estimates of type (25) and (26) for the functions \((p_{n-1}^*, \theta_{n-1}, w_{n-1})\) were obtained. For the sake of simplicity we rewrite Problem n.1 in the new notation.
Problem n.1 - Find \((p_n^*, \theta_n) \in M_0 \times L^\infty(\Omega)\) such that

\[
\int_\Omega l^3 \nabla p_n^* \cdot \nabla \varphi \, dx\,dy = \int_\Omega l_\theta \frac{\partial \varphi}{\partial x} \, dx\,dy - p_a \int_\Omega l^3 \frac{\partial \varphi}{\partial y} \, dx\,dy \quad \forall \varphi \in M_0
\]  

\[p_n^* \geq -p_a y \]  

\[H(p_n^* + p_a y) \leq \theta_n \leq 1 \]  

where

\[l(x, y) = h(x) + w_{n-1}(x, y)\]  

and therefore

\[0 < k_0 \leq \| l \|_{C(\Omega)} \leq k_1\]  

with \(k_0\) and \(k_1\) constants independents of \(n\). The existence of solution for Equations (31)-(33) is obtained by means of the regularization of the function \(\theta\) and the consideration of the following regularized problem:

Find \(p_{\epsilon}^* \in M_0\) such that

\[
\int_\Omega l^3 \nabla p_{\epsilon}^* \cdot \nabla \varphi \, dx\,dy = \int_\Omega l H_{\epsilon}(p_{\epsilon}^* + p_a y) \frac{\partial \varphi}{\partial x} \, dx\,dy - p_a \int_\Omega l^3 \frac{\partial \varphi}{\partial y} \, dx\,dy \quad \forall \varphi \in M_0
\]  

\[p_{\epsilon}^* \geq -p_a y \]  

where the function \(H_{\epsilon}\) is an approach of the Heaviside graph defined by

\[H_{\epsilon}(t) = \begin{cases} 
1 & t > \epsilon \\
\frac{t}{\epsilon} & 0 \leq t \leq \epsilon \\
0 & t \leq 0
\end{cases}\]  

**Theorem 1** For \(q_{\epsilon}^* \in L^2(\Omega)\) given there exists a unique function \(p_{\epsilon}^* \in M_0\) that is a solution of the linear problem

\[
\int_\Omega l^3 \nabla p_{\epsilon}^* \cdot \nabla \varphi \, dx\,dy = \int_\Omega l H_{\epsilon}(q_{\epsilon}^* + p_a y) \frac{\partial \varphi}{\partial x} \, dx\,dy - p_a \int_\Omega l^3 \frac{\partial \varphi}{\partial y} \, dx\,dy \quad \forall \varphi \in M_0
\]  

**Proof**: It is a consequence of Lax-Milgram theorem.
Theorem 2 The regularized problem (36) – (37) has a unique solution.

Proof: The existence is achieved by means of Schauder fixed point theorem applied to the operator defined by the solution of the linear Equation (39) as in the work of Alvarez [1]. The uniqueness is based in the use of special functions (see Brezis-Kinderleher-Stampacchia [7] and Vázquez [14]) and the upper and lower bounds of l.

Theorem 3 The Problem n.1 has at least a solution.

Proof: From Equation (37) the function $p^*_e + p_a y$ is non negative and the set

$$M_0^+ = \{ \varphi \in M_0 / \varphi + p_a y \geq 0 \} \quad (40)$$

is weakly closed in $M_0$. Moreover $p^*_e \in M_0^+$.

Easy computations allow us to obtain $H^1(\Omega)$ estimates for $p^*_e$ independents of the parameter $\epsilon$. Thus, from compacity arguments we deduce that there exists $p^*_n \in M_0^+$ such that

$$\{ p^*_e \} \rightarrow p^*_n \text{ in } H^1(\Omega) \text{ weak} \quad (41)$$

where $\{ p^*_e \}$ is really a subsequence of $\{ p^*_e \}$. By the same argument there exists $\theta_n \in L^2(\Omega)$ such that

$$0 \leq \theta_n \leq 1 \quad \text{and} \quad H_\epsilon(p^*_e + p_a y) \rightarrow \theta_n \text{ weak} \quad (42)$$

and the convergences are in the $L^2(\Omega)$ weak and $L^\infty(\Omega)$ weak-* topologies. The convergences achieved allow to pass to the limit in the parameter $\epsilon$ in the regularized problem and conclude that $(p^*_n, \theta_n)$ is a solution of Problem 1.

Proposition 1 The function $p^*_n$ is continuous in $\Omega$.

Proof: The continuity in $\Omega$ is a classical result of regularity properties for solutions of second order elliptic problems. The continuity in the boundary follows from an argument analogous to the one used in Alvarez [1] (see Vázquez [14], for details).

With respect to Problem n.2 the existence and uniqueness of solution $w_n$ is a classical result for the biharmonic operator (see Rektoris [12], for example). Moreover taking into account that $p^*_n$ is non negative the weak maximum principle implies that $w_n$ is also non negative. The regularity property $w_n \in H^3(\Omega)$ (and therefore $w_n \in C^1(\Omega)$) is obtained in Cimatti [8]).

Finally the analogous estimates of step 1 for $p_n$ and $w_n$ are obtained:

$$\| p_n \|_{H^1(\Omega)} \leq K_3(\Omega, \beta, C) \quad (43)$$

$$\| w_n \|_{H^2(\Omega)} \leq K_4(\Omega, \beta, C) \quad (44)$$
where $K_3(\Omega, \beta, C)$ and $K_4(\Omega, \beta, C)$ denote constants independent of $n$.

The following theorem concludes the existence of solution for the problem here posed as the limit of the sequence that has been built up by means of the algorithm.

**Theorem 4** The problem (12) – (15) has at least a solution.

**Proof:** The previous estimates and the compacity arguments imply the convergences

\[
\exists p^* \in M_0 \mid p^*_n \rightharpoonup p^* \text{ in } H^1(\Omega) \text{ weak}
\]

\[
\exists w \in L_0 \mid w_n \rightharpoonup w \text{ in } H^2(\Omega) \text{ weak}
\]

\[
\exists \theta_n \in L^\infty(\Omega) \mid \theta_n \rightharpoonup \theta \text{ in } L^2(\Omega) \text{ and in } L^\infty(\Omega) \text{ weak}
\]

and consequently

\[
\Delta w_n \rightharpoonup \Delta w \text{ in } L^2(\Omega) \text{ weak}
\]

\[
p^*_n \rightharpoonup p^* \text{ in } L^2(\Omega) \text{ weak}
\]

\[
\nabla p_n^* \rightharpoonup \nabla p^* \text{ in } L^2(\Omega) \text{ weak}
\]

\[
\theta_n \rightharpoonup \theta \text{ in } L^2(\Omega) \text{ weak}
\]

Finally by passing to the limit in $n$ in the Equations (27) and (30) it is deduced that the triple $(p^*, \theta, w)$ is a weak solution of the elastohydrodynamic problem of Equations (12)-(15) here treated.

**NUMERICAL SOLUTION**

For the numerical solution we propose to follow the theoretical algorithm which has been presented in the last paragraph in order to obtain the existence of solution. First we decouple the hydrodynamic and the elastic part (problem n.1 and n.2 respectively). For the lubrication subproblem n.1 we adopt the ideas already developed in Bermúdez-Durany [5] by introducing an "artificial evolutive" problem and the total derivative. The numerical approach of the linear fourth order Problem n.2 is based in a classical equivalent mixed formulation (see Glowinsky-Pironneau [11]) which essentially involves the solution of two second order linear elliptic problems:

\[
-\Delta \chi = p \text{ in } \Omega \tag{45}
\]

\[
\chi = 0 \text{ on } \Gamma_0 \cup \Gamma_a \text{ and } 2\pi - x \text{ periodic} \tag{46}
\]

\[
-\eta \Delta w = \chi \text{ in } \Omega \tag{47}
\]

\[
w = 0 \text{ on } \Gamma_0 \cup \Gamma_a \text{ and } 2\pi - x \text{ periodic} \tag{48}
\]
So the global numerical algorithm here proposed remains as follows

- Initialize \( p_0 = p_a y, \ w_0 = 0 \) and \( \theta_0 = 1 \).

- Step n.1.: Compute \((p_n, \theta_n)\) as solution of

\[
\int_{\Omega} (h + w_{n-1})^3 \nabla p_n \nabla \varphi \, dx \, dy + \int_{\Omega} (h + w_{n-1}) \theta_n \frac{D \varphi}{Dt} \, dx \, dy = 0 \ \forall \varphi \in M_0
\]  

(49)

\[
H(p_n) \leq \theta_n \leq 1
\]  

(50)

- Step n.2.: Compute \( w_n \) in two steps:

\[
\int_{\Omega} \nabla \chi_n \nabla \psi \, dx \, dy = \int_{\Omega} p_n \psi \, dx \, dy \ \forall \psi \in L_0
\]  

(51)

\[
\eta \int_{\Omega} \nabla w_n \nabla \psi \, dx \, dy = \int_{\Omega} \chi_n \psi \, dx \, dy \ \forall \psi \in L_0
\]  

(52)

where for the sake of simplicity a *-notation has been dropped in Equation (49) in all the "artificially time dependent" functions defined by

\[
\varphi_*(x, y, t) = \varphi(x, y)
\]  

(53)

After introducing the "artificial" velocity field \( v(x, y) = (-1, 0) \), the total derivative

\[
\frac{D \varphi_*}{Dt} = -\frac{\partial \varphi}{\partial x}
\]  

(54)

appearing in Equation (49) is approached by means of the method of characteristics (see Durany-Vázquez [9]) and the resulting nonlinear problem is solved by duality algorithms (see Bermúdez-Moreno [6]). A finite element technique of type \( P_1 \)-Lagrange triangular elements is used as space discretization in all linear problems. A more detailed explanation about the numerical solution and several test examples are presented in Durany-García-Vázquez [10].

References


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