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Stability and bifurcation of plane cracks of arbitrary shape

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ABSTRACT

The extension of a plane crack of arbitrary shape in an elastic solid is discussed for G-based laws of propagation. It is shown that the rate of extension is governed by a variational inequality in which the second derivative of the potential energy and of the dissipated energy play a fundamental role. Crack surface is the principal unknown, the differentiation of energy must be performed with respect to a variable domain with moving boundary. Bifurcation and stability of the crack front curve can be discussed as in plasticity. The obtained results are illustrated by some simple analytical examples.

INTRODUCTION

The propagation of plane cracks of arbitrary shape is an interesting problem in fatigue or in fracture analysis. For example, the study of a plane crack of delamination propagating in laminated composites, or of interface cracks in thin films or in surface coatings of different kind, has been the subject of many discussions in the recent literature [1,3,9,10,14,15]. On the other hand, some models of damage mechanics also lead to study the extension of a damage zone in a solid [2,4,6,7,12,14]. The objective of this paper is to present some general results on the subject concerning the rate problem and stability or bifurcation analysis.

GENERAL EQUATIONS

An elastic solid with a propagating plane crack is a mechanical system undergoing irreversible transformation. The associated irreversible parameter is the crack surface, a plane domain Ω of boundary S as shown in Fig. 1.
Its evolution is associated with a total potential energy W :

$$W = \int_V w(\epsilon(u)) \; dV - \int_{S_T} F(\lambda).u \; ds \tag{1}$$

In this expression, $w(\epsilon)$ denotes the volumic density of elastic deformation and F is the applied forces, assumed to depend on a force or displacement control parameter λ.
If only quasi-static evolutions are considered, it is well known that the displacement field at equilibrium **u** can be implicitly defined as a function of the given state of crack Ω and of control λ :

$$\mathbf{u} = \mathbf{u}(\Omega,\lambda) \tag{2}$$

via the equilibrium equations :

$$\int_V w_{,\epsilon} \; \delta\epsilon \; dV - \int_S F \; \delta u \; ds = 0 \; . \tag{3}$$

The total potential energy can be then considered as a function of Ω and λ :

$$W = W(\Omega,\lambda) \; . \tag{4}$$

To introduce the generalized force associated with an extension of Ω, cf. [4,12], it is necessary to make the derivation of energy with respect to a domain $\mathbf{W}_{,\Omega}$ by the techniques of derivation with respect to a geometric domain.

It is established that if $\delta\Omega$ denotes the rate of the normal extension of the present boundary S, then the following expression holds :

$$\mathbf{W}_{,\Omega} \cdot \delta\mathbf{\Omega} = - \int_S G \; \delta\Omega \; ds \tag{5}$$

where G denotes the energy release rate at a point of the moving surface S and represents the local value of generalized force **G** associated with the motion of $\mathbf{\Omega}$.

For example, for a plane crack in a three-dimensional solid, G is the limiting value of the local Rice-Eshelby integrals :

$$G = J^0 \quad \text{with} \quad J^0 = \mathrm{Lim}_{\Gamma \Rightarrow 0} \int_\Gamma (w n_1 - n.\sigma.u_{,n}) \; d\Gamma \tag{6}$$

For a damage zone, the expression of G is [12] :

$$G = [w - n.\sigma.u_{,n}] \tag{7}$$

and for a delamination crack in a composite plate or a thin film [3,14,15] :

$$G = [w - n.N.u_{,n} - n.M.\nabla\nabla w.n] \tag{8}$$

From (5) and from the energy balance, it follows that the dissipation (which is also the product of the entropy production by the temperature) is simply a product of forces and fluxes :

$$d = \mathbf{G} \cdot \dot{\mathbf{\Omega}} = \int_S G(s) \cdot \dot{\Omega}(s) \; ds \qquad (9)$$

It may be useful to remark that in the same spirit, the second derivative of energy is :

$$\delta\mathbf{\Omega} \cdot \mathbf{W},_{\Omega\Omega} \cdot \delta\mathbf{\Omega} = \int_S - \left[\delta G + \delta\Omega \frac{G}{R} \right] \cdot \delta\Omega \; ds \qquad (10)$$

where δG denotes the variation of G following the motion $\delta\Omega$ of Ω and R is the local curvature of S. A more symmetric expression of the second derivative can be obtained from the expression of $\delta\mathbf{G}$ in terms of $\delta\mathbf{\Omega}$.

The G-based crack propagation law :

$$\begin{array}{ll} \text{If } G(s) < G_c \text{ then } \dot{\Omega}(s) = 0 & \text{(no propagation)} \\ \text{If } G(s) = G_c \text{ then } \dot{\Omega}(s) \geq 0 & \text{(possible propagation)} \end{array} \qquad (11)$$

is associated with the dissipation potential :

$$\mathbf{D}(\dot{\mathbf{\Omega}},\mathbf{\Omega}) = \int_S G_c \; \dot{\Omega}(s) \; ds \quad \text{for } \dot{\Omega}(s) \geq 0 \qquad (12)$$

If G_c is a constant, the total mechanical energy $\mathbf{\Phi}$ can be introduced as a function of the present state :

$$\mathbf{\Phi}(\mathbf{\Omega},\lambda) = \mathbf{W}(\mathbf{\Omega},\lambda) + G_c \int_\Omega da \qquad (13)$$

where the second term is the total potential energy and the third term represents the surface energy dissipated by crack extension.

RATE PROBLEM

The rate problem of propagation of the crack surface Ω consists in the obtention of the normal extension rate $\dot{\mathbf{\Omega}}$ in terms of the control rate $\dot{\lambda}$ when the present state is assumed to be known. Local rate equations follow directly from (11). After time derivation, the identity $(G(s) - G_c)\,\dot{\Omega}(s) = 0$ leads to :

$$\dot{\Omega}(s) \geq 0 \text{ if } G(s) = G_c \text{ and if } \frac{dG}{dt}(s) = 0 \qquad (14)$$

$$\dot{\Omega}(s) = 0 \text{ if } G(s) < G_c \text{ or if } G(s) = G_c \text{ but } \frac{dG}{dt}(s) < 0 .$$

where $\frac{dG}{dt}$ denotes the normal derivative of G following the motion of the boundary. These equations can also be written in an equivalent variational form. Indeed, after (14) :

$$\dot{\Omega}(s) \geq 0, \quad \frac{dG}{dt}(s) \leq 0 \text{ and } \frac{dG}{dt} \cdot \dot{\Omega} = 0 \text{ if } G(s) = G_c , \qquad (15)$$

it follows that :

$$\frac{dG}{dt}(s) \; (\delta\Omega - \dot{\Omega}(s)) \; \geq 0 \quad \text{for all } \delta\Omega \geq 0 \tag{16}$$

or, in a global way :

$$\int_{S_c} \frac{dG}{dt}(s) \; (\delta\Omega(s) - \dot{\Omega}(s)) \; ds \quad \geq 0 \;\; \text{for all admissible } \boldsymbol{\delta\Omega}. \tag{17}$$

Admissible rates $\boldsymbol{\delta\Omega}$ must satisfy $\delta\Omega(s) \geq 0$ on the portion S_c where the propagation limit is attained $G(s) = G_c$ and $\delta\Omega(s) = 0$ if $G(s) < G_c$.

Formally, from (5), (10) , (14) and (17), the rate $\dot{\boldsymbol{\Omega}}$ is then a solution of the following variational inequality :

$$\begin{gathered} \dot{\Omega}(s) \;\geq 0 \;\text{ on } S_c \;\text{ and satisfies } \forall\; \delta\Omega(s) \;\geq 0 \;\text{ on } S_c : \\ (\boldsymbol{\delta\Omega} - \dot{\boldsymbol{\Omega}}) \,.\, (\boldsymbol{\Phi}_{,\Omega\Omega} \,.\, \dot{\boldsymbol{\Omega}} \;+\; \mathbf{W}_{,\Omega\lambda} \,.\, \dot{\boldsymbol{\lambda}}) \;\geq 0 \end{gathered} \tag{18}$$

where the second derivative $\Phi_{,\Omega\Omega}$ plays a fundamental role.

BIFURCATION AND STABILITY ANALYSES

As in incremental plasticity, the study of the rate problem enables us to follow step by step the evolution of the crack surface. For example, stability and bifurcation of the quasi-static response can be discussed as in plasticity by Hill's method [8,12] . The following propositions are then obtained :

The present equilibrium is stable in the dynamic sense if :

$$\boldsymbol{\delta\Omega} \,.\, \boldsymbol{\Phi}_{,\Omega\Omega} \,.\, \boldsymbol{\delta\Omega} \;\text{ is positive definite for } \delta\Omega(s) \geq 0 \text{ on } S_c . \tag{19}$$

The stability criterion (19) can also be written as :

$$\int_{S} - \delta G \,.\, \delta\Omega \;\, ds \;\; > 0 \quad \text{for any } \boldsymbol{\delta\Omega} \neq \mathbf{0} \text{ such that } \delta\Omega(s) \geq 0 \text{ on } S_c . \tag{20}$$

The present equilibrium is not a bifurcation point if the rate response is unique. Since uniqueness is ensured by a similar but more restrictive positive condition by relaxing the sign of $\delta\Omega(s)$ on S_c :

$$\boldsymbol{\delta\Omega} \,.\, \boldsymbol{\Phi}_{,\Omega\Omega} \,.\, \boldsymbol{\delta\Omega} \;\text{ is a positive definite for all } \boldsymbol{\delta\Omega} , \tag{21}$$

condition (21) represents a sufficient condition of non-bifurcation.

ILLUSTRATION

As an illustration, consider the debonding of a thin film which represents the surface coating of a rigid half-space, due to the propagation of an interface crack Ω with internal pressure p, cf. Fig. 2 . The film is assumed to be a membrane in isotrope tension T. If u is the transverse displacement of the membrane at point x, the associated elastic energy is $w = \frac{1}{2} T |\nabla u|^2$.

If the pressure p is controlled, $p = \lambda$ and the total potential energy is :

$$W = \int_\Omega \frac{1}{2} T |\nabla u|^2 \, da - \int_\Omega \lambda u \, da . \tag{22}$$

Local equations at equilibrium are :

$$T \Delta u + p = 0 \text{ in } \Omega , \quad u = 0 \text{ on } S . \tag{23}$$

Since $W_{,\Omega} . \delta\Omega = \int_\Omega (T \nabla u . \nabla \delta u - \lambda \delta u) \, da + \int_S (w - \lambda u) \, \delta\Omega \, ds$

where δu is associated to $\delta\Omega$ by the perturbation boundary problem which follows from (23) :

$$T \Delta \delta u = 0 \text{ in } \Omega , \ \delta u + \nabla u.n \, \delta\Omega = 0 \text{ on } S , \tag{24}$$

finally one obtains :

$$W_{,\Omega} . \delta\Omega = - \int_S w \, \delta\Omega \, ds . \tag{25}$$

Thus $G(s) = w = \frac{1}{2} T |\nabla u(s)|^2$ and $\delta G = T \nabla u . \nabla \delta u + T \nabla u . \nabla\nabla u . n \, \delta\Omega$, the quadratic form to be considered is :

$$\boldsymbol{\delta\Omega . \Phi_{,\Omega\Omega} . \delta\Omega} = \int_S -\delta G . \delta\Omega \, ds = \int_S - T \nabla u . (\nabla \delta u + \nabla\nabla u.n \, \delta\Omega) \, \delta\Omega \, ds . \tag{26}$$

The last expression can also be written in a symmetric form as :

$$\int_\Omega T |\nabla \delta u|^2 \, da + \sqrt{2TG_c} \int_S u_{,nn} \, \delta\Omega^2 \, ds .$$

If the internal volume is controlled, i.e. if :

$$\int_\Omega u \, da = \lambda , \tag{27}$$

the total potential energy is also given by the expression of the lagrangean :

$$W(\Omega,\lambda) = \int_\Omega w \, da - p \left[\int_\Omega u \, da - \lambda \right]$$

p is the lagrangean multiplier associated with the volume constraint (27) ; p and u are implicitely defined by (23) and (27).

Thus $W_{,\Omega} \,.\, \delta\Omega \;=\; \int_\Omega (T\, \nabla u\, \nabla \delta u - p\, \delta u)\, da + \int_S (w - pu)\, \delta\Omega\, ds$

with $T\, \Delta\delta u \;+\; \delta p \;= O \text{ in } \Omega\,,\;\; \delta u + \nabla u.n\, \delta\Omega \;= O \text{ on } S\,,\;\; \int_\Omega \delta u\, da \;= 0\,.$ (28)

Finally, expressions (25), (26) still hold with a different definition of δu.

Consider for example the case of a circular interface crack of radius R, cf. Bérest [1]. Equations (23) give $u(r,\theta) \;=\; \frac{p}{4T}\,(R^2 - r^2)$. The propagation limit G_c is attained on the whole contour S when $p = p_c = \frac{2}{R}\sqrt{2TG_c}$ which is a limit value since equilibrium is not possible for $p > p_c$. Let us study the stability of the equilibrium when $p = p_c$:

A boundary extension rate $\delta\Omega(\theta)$ can be expanded in Fourier series :

$$\delta\Omega(\theta) \;=\; \delta a_0 + \sum_1^\infty (\delta a_j \cos j\theta + \delta b_j \sin j\theta)\,. \tag{29}$$

The associated rate δu defined by (24) is :

$$\delta u(r,\theta) \;=\; \frac{pR}{2T}\left\{ \delta a_0 + \sum_1^\infty (\delta a_j \cos j\theta + \delta b_j \sin j\theta) \left(\frac{r}{R}\right)^j \right\}$$

Relation (26) leads to :

$$\delta\Omega \,.\, \Phi_{,\Omega\Omega} \,.\, \delta\Omega \;=\; 2\pi\, G_c \left[-2\, \delta a_0^{\,2} + \sum_1^\infty (j-1)\, (\delta a_j^{\,2} + \delta b_j^{\,2}) \right]$$

Thus, the considered equilibrium is unstable in mode 0 with pressure control.

In volume control, a similar result is obtained :

$$\delta\Omega \,.\, \Phi_{,\Omega\Omega} \,.\, \delta\Omega \;=\; 2\pi\, G_c \left[6\, \delta a_0^{\,2} + \sum_1^\infty (j-1)(\delta a_j^{\,2} + \delta b_j^{\,2}) \right]$$

It is not difficult to check that (20) is satisfied while (21) is not. The considered equilibrium is stable but bifurcation is always possible in mode 1 since the boundary extension rate can be of the form $\delta\Omega(\theta) \;=\; \frac{1}{3}\, R\, \frac{\dot\lambda}{\lambda} \;+\; \dot a_1 \cos\theta + \dot b_1 \sin\theta$, where $\dot a_1$ and $\dot b_1$ are arbitrary small numbers such that $\delta\Omega(\theta)$ is non-negative.

The tunnel crack, cf. Fig. 2b where Ω is an infinite band of width 2R, is also an interesting example. Transverse displacement is now $u \;=\; \frac{p}{2T}\,(R^2 - x^2)$ after (23).

A symmetric mode of bifurcation of the crack front of the form :

$\delta\Omega(y) = \delta a + \delta b \cos ky$ when $x = R$, $\delta\Omega(y) = \delta a + \delta b \cos ky$ when $x = -R$ (30)

can be considered. In this case, it follows from (28) that :

$$\delta u = \frac{\delta p}{2T}(R^2-x^2) + \frac{pR}{T}\delta a + \delta b \frac{pR}{T}\frac{\mathrm{ch}kx}{\mathrm{ch}kR}\cos ky \text{ with } \int_\Omega \delta u \, da = 0 \text{ thus } \delta p$$

$= -\delta a \frac{3p}{R}$ in volume control. By unit length, the quadratic form (26) is :

$$\boldsymbol{\delta\Omega} \cdot \boldsymbol{\Phi}_{,\Omega\Omega} \cdot \boldsymbol{\delta\Omega} = 10\, G_c \frac{1}{R}\delta a^2 + \delta b^2 G_c \frac{1}{2R}(kR \,\mathrm{th}kR - 1) \text{ with } kL = 2\pi$$

Thus, a symmetric bifurcation of the front following a sinus curve of wave length L, with $\frac{2\pi R}{L} \,\mathrm{th}\, \frac{2\pi R}{L} = 1$, is always possible.

An skew-symmetric mode of bifurcation of the form :

$\delta\Omega(y) = \delta a + \delta b \cos ky$ when $x = R$, $\delta\Omega = -\delta a - \delta b \cos ky$ when $x = -R$ (31)

can also be considered. In this case :

$$\delta u = \frac{\delta p}{2T}(R^2-x^2) + \frac{pR}{T}\delta a \frac{x}{R} + \frac{pR}{T}\delta b \frac{\mathrm{sh}kx}{\mathrm{sh}kR}\cos ky \text{ with } \int_\Omega \delta u \, da = 0 \text{ thus } \delta p$$

$= 0$ in volume control. By unit length, the quadratic form (26) is :

$$\boldsymbol{\delta\Omega} \cdot \boldsymbol{\Phi}_{,\Omega\Omega} \cdot \boldsymbol{\delta\Omega} = G_c \frac{1}{2R}\delta b^2 (kR \coth kR - 1)$$

A skew-symmetric bifurcation is always possible following a translation mode (arbitrary δa) or a sinus mode of wave length L with $\frac{2\pi R}{L} \coth \frac{2\pi R}{L} = 1$.

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Fig.1 Propagation of a plane crack

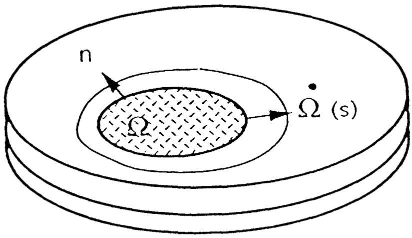

Fig.2 Thin membrane under internal pressure
2a : Circular crack
2b : Tunnel crack

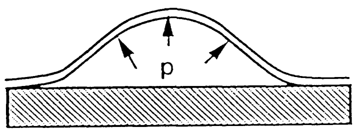

Fig. 2a

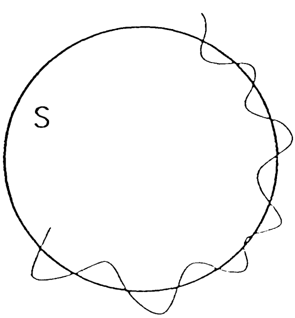

Fig. 2b