A boundary collocation heat flow model for arc welding

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ABSTRACT

Heat flow during welding may strongly affect phase transformation in the workpiece material. Therefore, there will be possible changes in physical as well as in mechanical properties of the material. For these reasons, the welding process requires a suitable computational method for estimating peak temperature and also the cooling rate during welding. Analytical models may be found for idealized models involving infinitely large regions, simple boundary conditions and constant material properties. Consequently, general cases involving finite geometries, temperature-dependent material properties and arbitrary boundary conditions may require a suitable numerical method. In this paper different governing differential equations of welding are investigated. In general, the resulting governing differential equations are nonlinear and their numerical solution may require an iterative approach. A boundary-oriented formulation based on the fundamental collocation [1] is presented which is suitable for use on personal computers. The method is illustrated using an example involving temperature distribution in a thin plate during arc welding.

INTRODUCTION

Welding is a commonly used fabrication process which has gained attention in recent years because of its heat transfer phenomena. An analytical model based on idealized assumptions is available [2] which may be used to estimate temperature distribution as well as cooling rates. Modifications including a Gaussian distribution of the heat source are given in [3,4]. However, since material properties change at higher temperatures, a numerical procedure such as the finite element method [5] becomes necessary. Unfortunately, the finite element method for time-dependent problems and/or temperature-dependent properties becomes tedious and may not be suitable on personal computers. In this paper, an alternate approach based on the boundary element method is presented which does not require extensive computation or computer memory. The method can be used to find temperature distributions as well as cooling rates in the workpiece.

GOVERNING DIFFERENTIAL EQUATIONS OF ARC WELDING

The governing differential equation of heat flow in an arbitrary region R during welding may be obtained from the energy equation. In the absence of radiation, Joule heating, heat generation (or consumption), and viscous dissipation, the energy equation for a volume element fixed in the body (i.e., material element) inside workpiece R, (**Figure 1**) becomes:

$$\nabla . K \nabla T = \rho C_p \partial T / \partial \tau \qquad \text{inside R} \qquad (1)$$

where T is temperature, K is the thermal conductivity of the workpiece material, ρ is density, C_p is specific heat, τ is time and ∇ is the gradient operator in the fixed coordinates (X,Y,Z). On

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the boundary of the region, ∂R, a Rubin-type boundary condition is considered, i.e.,

$$aT + b\,\partial T/\partial n = c \qquad \text{on } \partial R \tag{2}$$

where n is the outward normal to the boundary and c is the prescribed boundary condition on ∂R. Parameters a and b are selected as follows:

$$\begin{array}{ll} \text{For the Dirichlet condition} & : a=1,\ b=0 \\ \text{For the Neumann condition} & : a=0,\ b=1 \\ \text{For Rubin's-type condition} & : a\neq 0,\ b\neq 0 \end{array} \tag{3}$$

The governing equation (1) corresponds to the stationary (fixed) coordinate system (X,Y,Z) in which differentiations are performed in the (X,Y,Z) coordinates and $T=T(X,Y,Z;\tau)$. With respect to the moving coordinate system (x,y,z), which is attached to the electrode (heat source), the solution is expressed as $T=T(x,y,z;t)$. In this case the energy equation applies to a volume of a material at a particular (x,y,z) location which is not a volume fixed in the body. If the coordinate system moves past the volume element at a speed u along the x-axis we can find, from the energy equation,

$$\nabla.K\nabla T = \rho C_p \Delta T/\Delta\tau \qquad \text{inside R} \tag{4}$$

Where ∇ is the gradient operator in the moving coordinates (x,y,z) coordinates and $\Delta T/\Delta\tau$ means change in temperature over time Δt for a fixed material element. Now, since the fixed element lies at $X=x+u\tau$, then since X=constant, $dX=0=dx+ud\tau$. This means that in order to be at a fixed location in the material, if one changes time by $d\tau$, then one must move back along the x-axis by an amount $dx=-ud\tau$ in order to stay at one spot in space. means

$$\Delta T/\Delta\tau = [T(x-u\Delta t,y,z;t+\Delta t)-T(x,y,z;t)]/\Delta t = -u\partial T/\partial x + \partial T/\partial t \tag{5}$$

Therefore, with respect to the moving coordinate system (x,y,z) the governing differential equation will be:

$$\nabla.K\nabla T = \rho C_p(-u\partial T/\partial x + \partial T/\partial t) \qquad \text{inside R} \tag{6}$$

Derivation of (6) by mathematical transformation of (1) is as follows: $\partial T/\partial X$ means $\Delta T/\Delta X$ at a fixed time. But at a fixed time, form $X=x+ut$ we obtain $\Delta X=\Delta x$. Therefore, $\partial T/\partial X=\partial T/\partial x$. Likewise for y and z. Therefore, the gradient operators in coordinate systems (X,Y,Z) and (x,y,z) are the same. However, $\partial T/\partial\tau$ means $\Delta T/\Delta\tau$ at a fixed (X,Y,Z). But since the solution is now expressed in terms of (x,y,z;t), fixing X and changing time requires that x changes. As in (5),

$$\Delta T = [T(x-u\Delta t,y,z;t+\Delta t)-T(x,y,z;t)] = (-u\Delta t)\partial T/\partial x + (\Delta t)\partial T/\partial t \tag{7}$$

we find that $\partial T/\partial\tau$ transforms to $-u\partial T/\partial x + \partial T/\partial t$.

Note that above results are easily obtained using the following mathematical chain rule:

$$\partial T/\partial\tau = (\partial T/\partial x)(\partial x/\partial\tau) + (\partial T/\partial t)(\partial t/\partial\tau) \tag{8}$$

Sincere $\partial x/\partial\tau=-u$ and $\partial t/\partial\tau=1$ we obtain

$$\partial T/\partial \tau = -u(\partial T/\partial x) + (\partial T/\partial t) \tag{9}$$

However, this method does not provide any physical interpretation.

COMPARISON OF (1) AND (6)

The governing differential equation (1) involves both time (τ) and space coordinates (X,Y,Z). The corresponding boundary condition(s) may become complicated because it involves a surface point heat source which moves from point to point on the surface. The governing equation (6), on the other hand, is complicated (in form) but its boundary condition is relatively simple; it involves a fixed point heat source. Furthermore, using the "*quasistationary*" assumption, $\partial T/\partial t = 0$, we can write

$$\nabla . K \nabla T = -\rho C_p u \partial T/\partial x \tag{10}$$

This assumption indicates that if the electrode moves at a constant speed and the thermal disturbance does not move faster than the electrode, then the temperature field appears to be stationary. The situation is similar to what happens in a boat: if the boat (electrode) moves faster than the speed of the waves (thermal disturbances) in the water, then the wake behind the boat at a certain distance behind always looks the same to an observer on the coordinates (x,y,z) on the boat. But if the boat moves slower than the waves, the disturbance (wake) propagate past the boat and the wake does not appear stationary.

EFFECTS OF MATERIAL PROPERTIES

Table 1 shows corresponding governing differential equations for three possible cases. In the isotropic case, material properties are considered to be independent of direction. In the orthotropic case, it is assumed that x-y are the axes of material symmetry. The temperature-dependent conductivity case is a more realistic model because of variation of K,C_p and α with temperature.

Table 1. Effects of the Material Behavior on the Governing Equation

Material	Properties	Governing Differential Equation
Isotropic	K=constant	$\nabla^2 T = -2u(\partial T/\partial x)/\alpha$
Orthotopic	$K=(K_x,K_y,K_z)$	$K_x(\partial^2 T/\partial x^2)+K_y(\partial^2 T/\partial y^2)+K_z(\partial^2 T/\partial z^2)= -2\rho C_p u(\partial T/\partial x)$
Temperature-Dependent Conductivity	K=K(T)	$\nabla^2 T=-\{(\partial K/\partial T)[(\partial T/\partial x)^2+(\partial T/\partial y)^2+(\partial T/\partial z)^2+2\rho C_p u(\partial T/\partial x)\}/K$

where $\alpha=\rho C_p/K$ is the thermal diffusivity of the workpiece material and $\nabla^2 T =(\partial^2 T/\partial x^2)+(\partial^2 T/\partial y^2)+(\partial^2 T/\partial z^2)$ (11)

The governing differential equation of the isotropic case and the K=K(T) case are special cases of the following equation:

$$\nabla^2 T = f(x,y,z,T,\partial T/\partial x,\partial T/\partial y,\partial T/\partial z) \tag{12}$$

The orthotopic governing differential equation may be converted to the form in Equation (12) by means of the following transformations:

$$x_1 = x/\sqrt{K_x}\ ,\ x_2 = y/\sqrt{K_y} \text{ and } z_2 = z/\sqrt{K_z} \tag{13}$$

In the K=K(T) case the resulting governing differential equation is nonlinear. As in many nonlinear cases, its numerical treatment may require an iterative approach.

TWO DIMENSIONAL MODELS

Although a three-dimensional model may be required for most welding processes, it is possible to study the problem by means of appropriate two dimensional models. If the plate thickness is small compared to other dimensions (**Figure 2**), then $\partial T/\partial z$ is negligible. and we can write (12) as

$$\nabla^2 T = f(x,y,T,\partial T/\partial x,\partial T/\partial y) \tag{14}$$

Consequently, the upper half of the top plane may be used for numerical purposes. A different two-dimensional model (suitable for thick plates) is a section in the y-z plane. However, since the heat source moves in the x-direction, the resulting governing differential equation for this case is (1) in the y-z plane. Clearly, in this case discretization of time becomes necessary. Discussion of this case and extension of the fundamental collocation approach to three dimensional case are subjects of future investigations.

NUMERICAL SOLUTION OF (14)

If T^* is a fundamental solution of $\nabla^2 T(x,y)=0$, then temperature T at any internal point F(xF,yF) may be expressed as

$$T(F) = \int_{\partial R} W(S)T^*(F;S)d\Gamma \ + \ \int_R f(x,y,T,\partial T/\partial x,\partial T/\partial y)T^*(F;P)dR \tag{15}$$

where $d\Gamma$ is the element along the boundary ∂R, dR is the surface element in R, S(xS,yS) and P(xP,yP) are integration points along ∂R and in R respectively. The basic fundamental solution of the Laplace equation in two dimensions is $T^* = \ln r^2$ where $r^2 = x^2 + y^2$ which satisfies $\nabla^2 T(x,y)=0$ for all (x,y) except (0,0). Consequently, a finite series of translated fundamental solutions in the form

$$\sum_{j=1}^{N} W_j \ln r^2(\mathbf{x};S_j) \tag{16}$$

$$\text{where } r^2(\mathbf{x};S_j) = (x - xS_j)^2 + (y - yS_j)^2 \tag{17}$$

also satisfies $\nabla^2 T(x,y)=0$ for any combination of N coefficients (weights), W_j, except at the "source" points $S_j = (xS,yS)_j$. Therefore, if it is required to satisfy $\nabla^2 T(x,y)=0$ inside the region, the source points need to be located outside R. For simplicity, the N sources are applied at source points $S_j = (xS,yS)_j$ located on a similar boundary at distance DS away from ∂R (**Figure 4**). The overall solution of Equation (15) which is the sum of the homogeneous part Equation (16) and the particular part (due to f), will be constructed by approximating the area integral using M cells of elemental area ΔA_k, k=1,2,3,...,M (**Figure 5**). If center points of these cells are represented by $P_k = (xP,yP)_k$ we can write

$$T(F) = \sum_{j=1}^{N} W_j \ln r^2(F;S_j) + (1/4\pi) \sum_{k=1}^{M} f_k \Delta A_k \ln r^2(F;P_k) \qquad (18)$$

where $r^2(F;S_j) = (xF - xS_j)^2 + (yF - yS_j)^2$ (19)

and $r^2(F;P_k) = (xF - xP_k)^2 + (yF - yP_k)^2$ (20)

When F and P_k coincide, the integration over a rectangular 2ax2b cell of area $\Delta A = 4ab$ is performed analytically. It can be shown that in this case

$$\int \ln(x^2+y^2)dA = [\ln(a^2+b^2) + (a^2+b^2)\pi/\Delta A - q\arctan q - 3]\Delta A \qquad (21)$$

where $q = .5(a/b - b/a)$ (22)

In a square cell, $a=b$, Equation (21) becomes $[\ln(\Delta A/2) + \pi/2 - 3]\Delta A$.

Since $\partial T/\partial n = 2(x\cos\theta + y\sin\theta)/r^2$ where θ is the angle that the normal to the boundary makes with the positive x-axis, we can simply obtain corresponding expressions for the Neumann and the Rubin-type boundary conditions. However, in the resulting expressions, as in Equation (14), the dependent variable T and its derivatives appear on both sides of the equation. Satisfying boundary conditions at boundary points $B_i(xB, yB)_i$ leads to an implicit solution to Equation (12). As with many implicit forms, the solution may be obtained in an iterative manner. The matrix representation for a general case will be

$$D\,w + G\,f = c \qquad (23)$$

where $D(d_{ij})$ $i=1,2,..,N$; $j=1,2,..,N$ is the influence matrix,
$w(w_i)$ $i=1,2,..,N$ is a vector containing the source strengths
$G(g_{ij})$ $i,j=1,2,...,N$; $j=1,2,..,M$ is the body force matrix
$f(fj)$ $j=1,..,M$ is a vector containing $\Delta A_k/(4\pi)$ times values of the forcing function f at field points F_j
$c(c_i)$ $i=1,2,..,N$ is a vector containing prescribed boundary conditions at points B_i.
For the Dirichlet condition : $d_{ij} = \ln r_{ij}2$ where $r_{ij} = r(B_i;S_j)$
For the Neumann condition : $d_{ij} = 2[(xB_i - xS_j)\cos\theta + (yB_i - yS_j)\sin\theta]/r_{ij}^2$
For the Rubin-type boundary condition a suitable combination of these (see Equation 2) is used.

THE ITERATION SCHEME

Multiplying Equation (23) by D^{-1} gives

$$w = p - E.f \qquad (23)$$

where $p = D^{-1}c$ and $E = D^{-1}G$

The unknowns are w_i and the f_j's are functions of w_i. Iteration starts with a guess $f^{(0)}$for f which is substituted in Equation (24) to compute $w^{(1)} = p - E.f^{(0)}$. Then $w^{(1)}$ is used to compute $f^{(1)}$ from which we compute $w^{(2)}$ and repeat the process in the hope that it converges. Once the w_i's are evaluated, temperature, T, and the temperature gradients, $\partial T/\partial x$ and $\partial T/\partial y$ are obtained from (18).

EXAMPLE

Consider the case where the corners of the upper half of the plate are symmetrically situated with respect to the electrode (i.e., the electrode is at $x=y=0$). Consequently, the first quadrant oABC would be appropriate for numerical solutions.

Workpiece material	: Steel AISI 1018
Melting point	: 1530 °C
Thermal Diffusivity	: $\alpha=.091$ cm^2/sec
Welding velocity	: $u=5$ mm/sec
Plate dimensions	: Length = 32 cm, width=8 cm, thickness=6 mm
Numerical Data	: $N=30$, $M=80$ (**Figures 4 & 5**)

Boundary Conditions: On oA: At the heat source $T=1530$ °C; otherwise $\partial T/\partial y=0$.
On oB & AC: $\partial T/\partial x=0$ On BC: T=Ambient temperature.

Using an initial guess of $f^{(0)}=0$ and DS=4 cm, the iterative scheme in this paper converged in about four iterations. The results are graphically shown in **Figures 6 and 7**. These results may be combined with the iron-carbon phase diagram to study phase transformation in the material. The cooling curve, $\partial T/\partial \tau$ versus x, may be constructed from temperature gradients according to

$$\partial T/\partial \tau=(\partial T/\partial X)(\partial X/\partial \tau)= u\partial T/\partial x \tag{24}$$

CONCLUSIONS

Various heat flow models for arc welding have been studied. A simple procedure based on the fundamental collocation method has been presented. The method utilizes an iterative process to handle a resulting nonlinear governing differential equation for arc welding.

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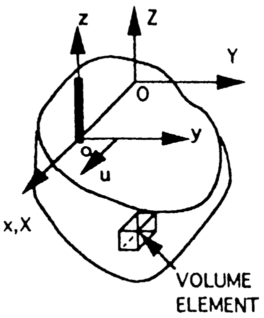

FIGURE 1. Schematic of Arc Welding (3D)

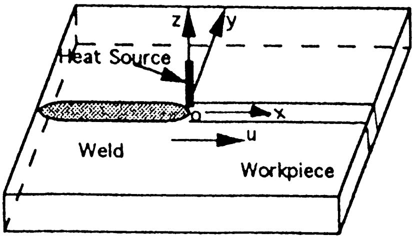

FIGURE 2. Two-Dimensional Model of Arc Welding

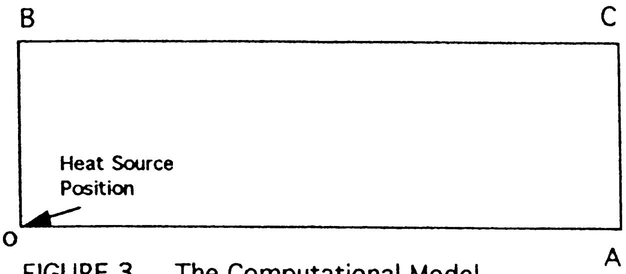

FIGURE 3. The Computational Model

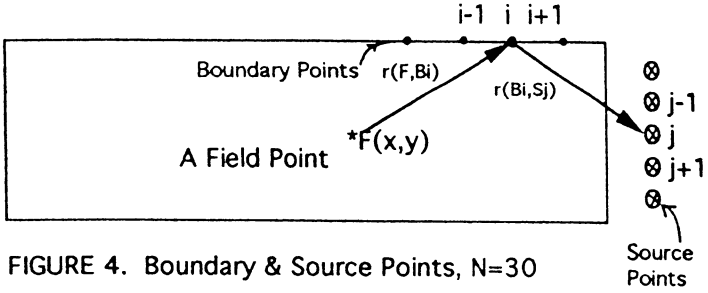

FIGURE 4. Boundary & Source Points, N=30

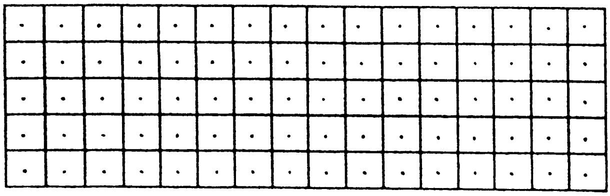

FIGURE 5. Internal Cells, M=80

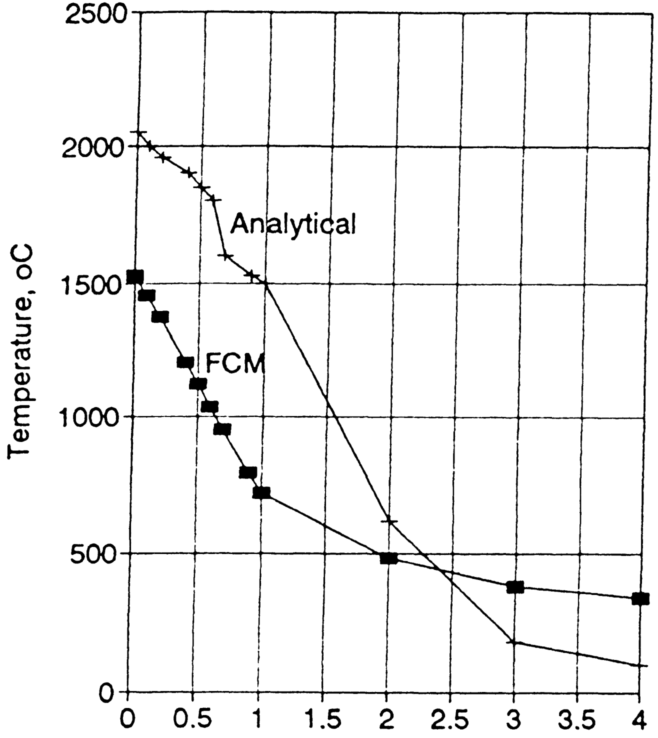

Fig. 6- Distance From Heat Source, cm

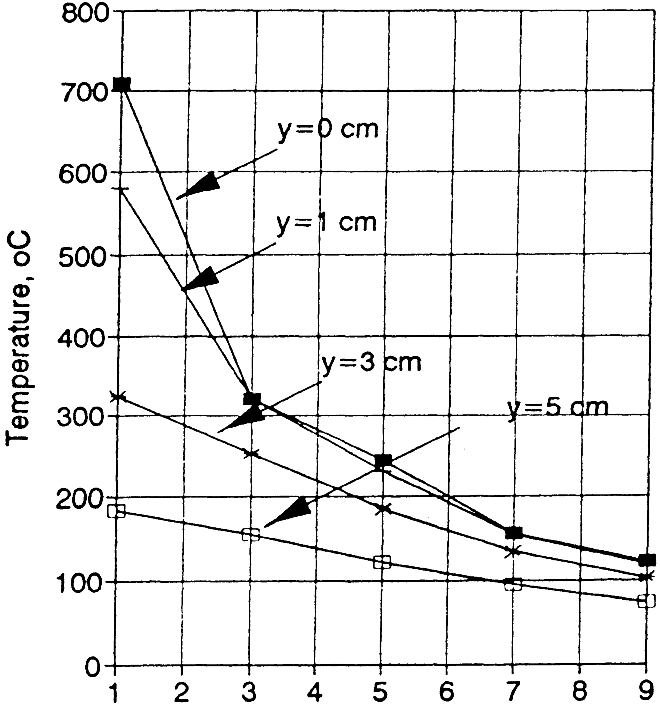

Fig. 7- Distance From Heat Source, cm