A polygon-free numerical solution of steady natural convection in solid-liquid systems

J. Perko¹, C.S. Chen², B. Šarler¹

¹Laboratory for Fluid Dynamics and Thermodynamics, Faculty of Mechanical Engineering, University of Ljubljana, Slovenia
²Department of Mathematical Sciences, University of Nevada, USA

Abstract

This paper describes a primitive variable formulated and Kansa’s method solved free boundary problem arising in the steady natural convection of the incompressible Newtonian solid-liquid phase change material. Solution of the coupled mass, momentum, and energy equations in two-dimensions is solved by using global multiquadrics approximation of the fields and global cubic spline approximation of the boundary geometry.

1 Introduction

Various aspects of science and technology are nowadays related to modelling of the solid-liquid phase change systems. A broad spectra of discrete approximative methods have been used in the past decades to evaluate them. In recent years, there is a strong development in meshless methods [1]. Among them, the collocation method using radial basis functions (RBFs), has emerged as a promising meshless technique. The RBFs have been originally used for scattered data approximation [2] and afterwards for solution of the PDEs [3]. The latter technique was named as Kansa’s method by Golberg & Chen [4]. The main advantage of the Kansa’s method is in its simplicity, applicability to different PDEs, and effectiveness in dealing with arbitrary dimension and complicated domains. The method has recently been successfully applied in many scientific and engineering disciplines [5, 6]. The Kansa’s method can be modified into Hermite interpolation scheme [7] for which the resulting collocation matrix is guaranteed to be nonsingular.
Both approaches give similar accuracy. This paper presents the solution of the free boundary problem arising in steady solid-liquid phase-change systems using Kansa’s method. The physically equivalent problem has been solved by the Dual Reciprocity Boundary Element Method (DRBEM) in our previous MBP conference paper [8] where the polygonization and integration over the boundary had to be employed. In the present method the integration as well polygonization over the boundary has been avoided. This fact might be particularly advantageous when solving solidification or melting problems, because they usually appear in complicated geometric arrangements. This paper shows basic implementation of the Kansa’s method for solving steady natural convection in single-phase fluid and phase change material. For this purpose the one-phase continuum formulation [9] has been used as a physical framework, and the Poisson formulation [10] of the general transport equation as the mathematical framework.

2 Governing equations

This paper deals with incompressible Newtonian solid-liquid phase change material confined to the domain \( \Omega \) with the boundary \( \Gamma \). Consider a connected fixed domain \( \Omega \) with boundary \( \Gamma \) occupied by a phase change material described with the temperature dependent desity \( \rho_P \) of the phase \( P \), temperature dependent specific heat at constant pressure \( c_p \), effective thermal conductivity \( k \), viscosity \( \mu \) and the specific latent heat of the solid-liquid phase change \( h_M \). The one-phase continuum formulation of the transport equations for the assumed system is

\[
\nabla \cdot \mathbf{v} = 0
\]

(1)

\[
\frac{\partial}{\partial t} (\rho h) + \nabla \cdot (\rho \mathbf{v} h) = \nabla \cdot (k \nabla T)
\]

(2)
with \( \mathbf{a} \) standing for the acceleration vector, \( \beta \) for the volumetric thermal expansion coefficient, and \( T_{\text{ref}} \) for the reference temperature. The second term in the body force is modelling the phase change, simply constituted through the phase change coefficient \( c_P \) (large number) and temperature dependent liquid fraction \( f_L \) which is assumed to vary as

\[
f_L(T) = \begin{cases} 
0 & T < T_S \\
(T - T_S)/(T_L - T_S) & T_S \leq T \leq T_L \\
1 & T > T_L
\end{cases}
\]  

where \( T_S \) stands for the solidus and \( T_L \) for the liquidus temperature. We seek for steady mixture temperature, velocity and pressure. The steady state solution is obtained as a limiting case of the transient solution with the timestep \( \Delta t \). The solution at time \( t_0 + \Delta t \) is obtained by assuming known temperature and velocity fields at time \( t_0 \) and boundary conditions of the Dirichlet, Neumann and Robin type, respectively.

3 Solution procedure

3.1 General formulation of the transport phenomena

For the purposes of the demonstration of the solution procedure, a transport phenomena problem can be briefly described in a general manner as the numerical solution of Eulerian transport equation, defined on a fixed domain \( \Omega \) with boundary \( \Gamma \), of the kind

\[
\frac{\partial}{\partial t} (\varrho \mathcal{F}(\Phi)) + \nabla \cdot (\varrho \mathbf{v} \mathcal{F}(\Phi)) = -\nabla \cdot (-\mathbf{D} \nabla \Phi) + S
\]

with \( \varrho, \Phi, t, \mathbf{v} \), and \( S \) standing for density, transport variable, time, velocity, source, and diffusion matrix

\[
\mathbf{D} \equiv \begin{pmatrix}
  d_{11} & d_{12} & d_{13} \\
  d_{21} & d_{22} & d_{23} \\
  d_{31} & d_{32} & d_{33}
\end{pmatrix}
\]

The dependent variable stands, for instance for the velocity component in each coordinate direction, or temperature, or the mass fraction of a chemical species. The function \( \mathcal{F} \) denotes the relation between the transported and the diffused variable such as for example relation between the enthalpy and the temperature.

The solution of the governing equation for the dependent variable at final time \( t = t_0 + \Delta t \) is sought, where \( t_0 \) represents the initial time and \( \Delta t \) the positive time increment. The solution is constructed by the initial and boundary conditions that follow. The initial value of transport variable \( \Phi(p, t_0) \) at a point with position vector \( p \) and time \( t_0 \) is defined through the known function \( \Phi_0 \)

\[
\Phi(p, t) = \Phi_0; \quad p \in \Omega \cup \Gamma
\]
The boundary $\Gamma$ is divided into connected or disjoint parts $\Gamma^D$, $\Gamma^N$ and $\Gamma^R$

$$\Gamma = \Gamma^D \cup \Gamma^N \cup \Gamma^R$$  \ (9)

with Dirichlet, Neumann and Robin type boundary conditions respectively. These boundary conditions are at the boundary point $p$ with normal $n_\Gamma$ and time $t_0 \leq t \leq t_0 + \Delta t$ defined through known functions $\Phi^D_p$, $\Phi^N_p$, $\Phi^R_p$ and $\Phi^R_{ref}$

$$\Phi = \Phi^D; \quad p \in \Gamma^D, \quad \frac{\partial}{\partial n_\Gamma}\Phi = \Phi^N; \quad p \in \Gamma^N$$

$$\frac{\partial}{\partial n_\Gamma}\Phi = \Phi^R (\Phi - \Phi^R_{ref}); \quad p \in \Gamma^R$$  \ (10)

The involved parameters of the governing equation and boundary conditions are assumed to depend on the transport variable, space and time. The solution procedure thus inherently involves iterations. The governing equation is transformed as follows. The diffusion matrix

$$D = D I + D'$$  \ (11)

is split into constant isotropic part $DI$, with $I$ denoting identity matrix, and the remaining nonlinear anisotropic part $D'$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad D' = \begin{bmatrix} d_{11} - D & d_{12} & d_{13} \\ d_{21} & d_{22} - D & d_{23} \\ d_{31} & d_{32} & d_{33} - D \end{bmatrix}$$  \ (12)

The transport equation is subsequently cast into Poisson form

$$\nabla^2 \Phi = \Theta$$  \ (13)

with

$$\theta = \left[ \frac{\partial}{\partial t}(\varphi \mathcal{F}(\Phi)) - S \right]/D, \quad \Theta = \left[ \varphi \mathbf{v} \mathcal{F}(\Phi) - D' \nabla \Phi \right]/D$$  \ (14)

The inhomogeneous terms are Taylor expanded as

$$\theta \approx \tilde{\theta} + \bar{\theta}_\phi (\Phi - \bar{\Phi}), \quad \Theta \approx \bar{\Theta} + \bar{\theta}_\phi (\Phi - \bar{\Phi})$$  \ (15)

with ‘bar’ denoting known value at previous iteration. The final form of the transformed equation becomes

$$\nabla^2 \Phi = \theta + \nabla \cdot \Theta = \tilde{\theta} + \bar{\theta}_\phi (\Phi - \bar{\Phi}) + \nabla \cdot \bar{\Theta} + \nabla \cdot \bar{\theta}_\phi (\Phi - \bar{\Phi})$$  \ (16)

### 3.2 Time discretization

The time discretization is made (for simplicity of presentation only) in a

$$\theta \approx \left[ \frac{\varphi \mathcal{F}(\Phi) - \varphi \mathcal{F}(\Phi_0)}{\Delta t} - S \right]/D$$  \ (17)

fully implicit (backward Euler) manner where subscript 0 represents the value at the initial time.
3.3 Kansa’s method

The unknown field $\Phi$ is approximated by the $N$ global approximation functions $\psi_n(p)$ and their coefficients $\zeta_n$

$$\Phi(p) \approx \psi_n(p) \zeta_n; \ n = 1, 2, \ldots N$$

(18)

The global radial basis function approximation is based on the multiquadrics with the free parameter $r_0$

$$\psi_n = (r_n^2 + r_0^2)^{1/2}$$

(19)

The coefficients can be calculated from the $N$ collocation equations of which $N_T$ are equally distributed over boundary $N_T$ and $N_Q$ over the domain $\Omega$. Let us define the boundary condition indicators in order to be able to represent the boundary collocation equations in a compact form. The Dirichlet, Neumann, and Robin type of boundary condition indicators $\chi^D$, $\chi^N$, and $\chi^R$ are

$$\chi^D(p) = \begin{cases} 1 & p \in \Gamma_D \\ 0 & p \notin \Gamma_D \end{cases}, \quad \chi^N(p) = \begin{cases} 1 & p \in \Gamma_N \\ 0 & p \notin \Gamma_N \end{cases}, \quad \chi^R(p) = \begin{cases} 1 & p \in \Gamma_R \\ 0 & p \notin \Gamma_R \end{cases}$$

(20)

The boundary collocation equations can be respectively written in the following form

$$[\chi^D_{1} \psi_{in} + \chi^N_{1} \frac{\partial}{\partial n_i} \psi_{in} + \chi^R_{1} \frac{\partial}{\partial n_i} \psi_{in}] \zeta_n$$

$$\approx \chi^D_{1} \Phi^D_{1} + \chi^N_{1} \Phi^N_{1} + \chi^R_{1} \Phi^R_{1} \{ \psi_{in} \zeta_n - \Phi_{ref} \}; i = 1, 2, \ldots N$$

(21)

The domain collocation equations can be written in the following form

$$[\nabla^2 \psi_{in} - (\hat{\theta}_i \Phi_{1} + \nabla \cdot \Theta_i \Phi_{1}) \psi_{in}] \zeta_n$$

$$= \tilde{\theta}_i + \nabla \cdot \tilde{\Theta}_i - \tilde{\theta}_i \Phi_{1} \Phi_{1} - \nabla \cdot \tilde{\Theta}_i \Phi_{1} \Phi_{1}; i = N_{\Gamma+1}, N_{\Gamma+2}, \ldots, N$$

(22)

The coefficients $\zeta_n$ are calculated from the following system of $N \times N$ linear equations

$$\Psi \zeta = b$$

(23)

with

$$\Psi_{in} = \chi^D_{1} \psi_{in} + \chi^N_{1} \frac{\partial}{\partial n_i} \psi_{in} + \chi^R_{1} \{ \frac{\partial}{\partial n_i} \psi_{in} - \Phi_{ref} \psi_{in} \};$$

$$i = 1, 2, \ldots, N_{\Gamma}, \ n = 1, 2, \ldots, N$$

$$\Psi_{in} = \nabla^2 \psi_{in} - \tilde{\theta}_i \Phi_{1} \psi_{in} - \nabla \cdot \tilde{\Theta}_i \Phi_{1} \psi_{in};$$

$$i = N_{\Gamma+1}, N_{\Gamma+2}, \ldots, N, \ n = 1, 2, \ldots, N$$

$$b_i = \chi^D_{1} \Phi^D_{1} + \chi^N_{1} \Phi^N_{1} - \chi^R_{1} \Phi^R_{1} \Phi_{ref}, \ i = 1, 2, \ldots, N$$

$$b_i = \tilde{\theta}_i + \nabla \cdot \tilde{\Theta}_i - \tilde{\theta}_i \Phi_{1} \Phi_{1} - \nabla \cdot \tilde{\Theta}_i \Phi_{1} \Phi_{1}; \ i = N_{\Gamma+1}, N_{\Gamma+2}, \ldots, N$$

(24)
3.4 Meshless representation of the boundary

The length $\ell_i$ of the contour between the boundary points $p_i$ and $p_{i-1}$ can be approximated by the Euclidean distance

$$\ell_i = \left[ (p_{x,i} - p_{x,i-1})^2 + (p_{y,i} - p_{y,i-1})^2 \right]^{1/2}$$  \hspace{1cm} (26)

with the cyclic condition $p_{i-1} = p_{N_r}; i = 1$. The total length of the boundary contour $\ell_\Gamma$ equals to

$$\ell_\Gamma = \ell_i \delta_{ij}; \quad i, j = 1, 2, \cdots, \Gamma_N$$  \hspace{1cm} (27)

The position of the boundary points can be represented by the meshless approximation with the parameter $\ell$

$$p_\xi(\ell) = \psi^\Gamma_n(\ell) \zeta^\Gamma_n; n = 1, 2, \cdots, N_{\Gamma+3}$$

$$\psi^\Gamma_n(\ell) = |\ell - \ell_n|^3$$

$$\psi^\Gamma_{N_r+1}(\ell) = 1 \quad \psi^\Gamma_{N_r+2}(\ell) = \ell \quad \psi^\Gamma_{N_r+3}(\ell) = \ell^2$$  \hspace{1cm} (28)

The following compatibility conditions are needed for determination of the coefficients $\zeta^\Gamma_n; n = N_{\Gamma+1}, N_{\Gamma+2}, N_{\Gamma+3}$

$$\psi^\Gamma_n(\ell_n) \zeta^\Gamma_n = \psi^\Gamma_n(0) \zeta_n \quad \frac{\partial}{\partial \ell} \psi^\Gamma_n(\ell_n) \zeta^\Gamma_n = \frac{\partial}{\partial \ell} \psi^\Gamma_n(0) \zeta_n$$

$$\frac{\partial^2}{\partial \ell^2} \psi^\Gamma_n(\ell_n) \zeta^\Gamma_n = \frac{\partial^2}{\partial \ell^2} \psi^\Gamma_n(0) \zeta_n$$  \hspace{1cm} (29)

The coefficients $\zeta^\Gamma_n$ are calculated from the following $N_{\Gamma+3} \times N_{\Gamma+3}$ system of linear equations

$$\Psi^\Gamma \zeta^\Gamma_\xi = b^\Gamma_\xi$$  \hspace{1cm} (30)

with

$$\psi^\Gamma_n = \psi^\Gamma_n; \quad i = 1, 2, \cdots, N_\Gamma; \quad n = 1, 2, \cdots, N_{\Gamma+3}$$

$$\Psi^\Gamma_n = \psi^\Gamma_n(\ell_n) - \psi^\Gamma_n(0); \quad i = N_{\Gamma+1}, \quad n = 1, 2, \cdots, N_{\Gamma+3}$$

$$\Psi^\Gamma_n = \frac{\partial}{\partial \ell} \psi^\Gamma_n(\ell_n) - \frac{\partial}{\partial \ell} \psi^\Gamma_n(0); \quad i = N_{\Gamma+2}, \quad n = 1, 2, \cdots, N_{\Gamma+3}$$

$$\Psi^\Gamma_n = \frac{\partial^2}{\partial \ell^2} \psi^\Gamma_n(\ell_n) - \frac{\partial^2}{\partial \ell^2} \psi^\Gamma_n(0); \quad i = N_{\Gamma+3}, \quad n = 1, 2, \cdots, N_{\Gamma+3}$$  \hspace{1cm} (31)

$$b^\Gamma_\xi = b^\Gamma_\xi; \quad i = 1, 2, \cdots, N_{\Gamma}$$

$$b^\Gamma_\xi = 0; \quad i = N_{\Gamma+1}, N_{\Gamma+2}, N_{\Gamma+3}$$  \hspace{1cm} (32)

The components of the normal on the boundary can be explicitly calculated

$$n_x(\ell) = \frac{\partial p_y}{\partial \ell}(\ell) \left( \frac{\partial p_x}{\partial \ell}(\ell)^2 + \frac{\partial p_y}{\partial \ell}(\ell)^2 \right)^{-1/2}$$

$$n_y(\ell) = -\frac{\partial p_x}{\partial \ell}(\ell) \left( \frac{\partial p_x}{\partial \ell}(\ell)^2 + \frac{\partial p_y}{\partial \ell}(\ell)^2 \right)^{-1/2}$$

$$\frac{\partial p_x}{\partial \ell}(\ell) = \frac{\partial}{\partial \ell} \psi^\Gamma_n(\ell) \zeta^\Gamma_n$$  \hspace{1cm} (33)
3.5 Numerical implementation

After each solution of the scalar $\Phi$ a relaxation with coefficient $c_{rel}$ is made

$$\Phi = \Phi + c_{rel} (\Phi - \Phi)$$

and the timestep iterations are stopped when the criterion

$$\left( \frac{|\Phi_{avg} - |\Phi_{avg}|}{\Phi_{avg}} \right) < \epsilon_{itr}$$

is reached, where subscript avg represents the average gridpoint value. The criterion of reaching the steady state is

$$\left( \frac{|\Phi_{avg} - |\Phi_{avg}|}{\Phi_{avg}} \right) < \epsilon_{sts}.$$  

The three governing equations (pressure correction Poisson equation replaces the mass conservation equation) are in each timestep solved in an iterative bundle. Their representation within general transport equation context is shown in Appendix. First, new pressure field is solved, based on the old velocity, pressure and temperature fields. Momentum equation is solved afterwards, based on the old temperature field and new pressure field. Subsequently, pressure correction field is solved based on the new velocity field. The new velocity field is corrected through the pressure correction field. Finally new temperature field is calculated, based on the new velocity field.

4 Numerical examples

Two numerical examples are shown in the present text. They are represented in dimensional form because of the space constraints. Geometry is a square with dimension 1[m]. Upper and lower boundaries are insulated, the left boundary is subject to temperature $-0.5$[C] and the right boundary is subject to temperature $+0.5$[C]. Phase change coefficient is set to $0$[kg m$^{-3}$ s$^{-1}$] in case of single-phase fluid and to $10^5$[kg m$^{-3}$ s$^{-1}$] in case of phase change material. The solidus and the liquidus temperatures are

$$T_S = -0.01$$[C],  
$$T_S = +0.01$$[C].

Multiquadrics constant $r_0$ is set to 0.01. All other material properties except $\beta = 10^4$[1/K] are set to unit values. The square is discretized into 120 boundary nodes and 900 domain nodes that give 1020 meshpoints. Other parameters are $\Delta t = 0.005$[s], $c_{rel} = 1$, $\epsilon_{itr} = 0.001$, $\epsilon_{sts} = 0.01$. No flow has been assumed at the initial time and the temperature is distributed linearly from $+0.5$[C] to $-0.5$[C] from the left to the right boundary. Figures 1 and 2 show solution of the natural convection with the single-phase fluid and the phase change material, respectively. Solution in Figures 1 and 2 reach steady state after 214 and 351 iterations. A proper single-phase fluid numerical implementation is confirmed by matching with the classical [11] benchmark. A proper phase-change numerical implementation is confirmed by matching with the very recent results obtained by the finite volume method [12]. A systematic comparative study of these two methods will appear elsewhere. The approximation of the boundary of the square with the cubic splines introduces errors in the position of the normals as seen in Figures 1 and 2.
Figure 1: Isotherms and velocity vectors of the natural convection in a square cavity. Example with the single-phase fluid. Maximum velocity is 19.8 [m/s]. Note the meshless representation of the square geometry. The calculated normals to the boundary are denoted by the dashed line.

5 Conclusions

This preliminary study shows basic elements of the Kansa's method based computational modelling of the natural convection in phase change materials as required in numerous natural and technological systems. To our knowledge it represents the first attempt to solve a solid-liquid phase change system by such type of meshless approach. The use of the RBF approximation of the boundary geometry appears to be new as well. The main advantage of the method represents truly polygon-free discretization and simple numerical implementation. The disadvantage represents resulting full asymmetric algebraic systems of equations that might become ill-conditioned and are difficult to solve in an efficient way in large-scale problems. However, there are ways to overcome mentioned difficulties, such as the use of compactly supported radial basis functions, tackled in our ongoing research. The example represented in Figure 2 mimics (narrow phase change interval) the phase change of a pure substance, a problem that is usually difficult to solve with a fixed-grid technique. The Kansas's method however demonstrates stability even for much coarser discretization and flow.
intensity as used in the present paper. Due to the nature of polygon-free discretization, the current development is particularly attractive in solving geometrically complex systems. The successful application of the Kansa’s method to present highly non-linear and coupled problem promises a new era of "meshless computing technology" as experienced decades ago with other main-stream numerical methods.

![Isotherms and velocity vectors of the natural convection in a square cavity. Example with the phase change material. Maximum velocity is 11.3 [m/s].](image)

**Figure 2:** Isotherms and velocity vectors of the natural convection in a square cavity. Example with the phase change material. Maximum velocity is 11.3 [m/s].

**Acknowledgement**

Present paper represents a part of the projects (1) **COST-P3: Simulation of Physical Phenomena in Technological Applications** that forms a part of the Slovenian national research programme **Multiphase Systems** supported by the Slovenian Ministry of Education, Science and Sport, and (2) **COPERNIKUS ULTRAWAT** supported by the European Union. The second author thanks University of Nevada, Las Vegas for the support of Faculty Development Leave at the Laboratory for Fluid Dynamics and Thermodynamics, Faculty of Mechanical Engineering, University of Ljubljana during spring 2001.
References


Appendix: Definition of coefficients

\(N \times N\) systems of equations

**Energy transport**

<table>
<thead>
<tr>
<th>variable</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Phi)</td>
<td>(T)</td>
</tr>
<tr>
<td>(C(\Phi))</td>
<td>(h(T))</td>
</tr>
<tr>
<td>(D)</td>
<td>(k)</td>
</tr>
<tr>
<td>(D'_{\xi})</td>
<td>0</td>
</tr>
<tr>
<td>(S)</td>
<td>0</td>
</tr>
<tr>
<td>(\theta)</td>
<td>(\frac{\rho C(\Phi)}{(\Delta t k)} - \frac{\rho C(\Phi)}{\Delta t k})</td>
</tr>
<tr>
<td>(\theta_{,\phi})</td>
<td>(\frac{\rho (c_p \Phi + h_M f_{\xi} \phi)}{\Delta t k})</td>
</tr>
<tr>
<td>(\Theta_x)</td>
<td>(\frac{\rho v_x C(\Phi)}{})</td>
</tr>
<tr>
<td>(\Theta_y)</td>
<td>(\frac{\rho v_y C(\Phi)}{\Delta t k})</td>
</tr>
<tr>
<td>(\Theta_{x,\phi})</td>
<td>(\frac{\rho v_x (c_p \Phi + h_M f_{\xi} \phi)}{})</td>
</tr>
<tr>
<td>(\Theta_{y,\phi})</td>
<td>(\frac{\rho v_y (c_p \Phi + h_M f_{\xi} \phi)}{\Delta t k})</td>
</tr>
</tbody>
</table>

**X-Momentum transport**

<table>
<thead>
<tr>
<th>variable</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Phi)</td>
<td>(v_x)</td>
</tr>
<tr>
<td>(C(\Phi))</td>
<td>(v_x)</td>
</tr>
<tr>
<td>(D)</td>
<td>(\mu)</td>
</tr>
<tr>
<td>(D'_{\xi})</td>
<td>0</td>
</tr>
<tr>
<td>(S)</td>
<td>(-P_{ix} + \rho a_{ix} (1 - \beta (T - T_{ref})) - c_p (1 - f_{\xi}) \Phi)</td>
</tr>
<tr>
<td>(\theta)</td>
<td>(\frac{\rho \Phi}{(\Delta t \mu)} - \frac{\rho \Phi}{\Delta t \mu} - S/\mu)</td>
</tr>
<tr>
<td>(\theta_{,\phi})</td>
<td>(\frac{\rho}{(\Delta t \mu)} + \frac{c_p (1 - f_{\xi})}{\mu})</td>
</tr>
<tr>
<td>(\Theta_x)</td>
<td>(\frac{\rho \Phi^2}{\mu})</td>
</tr>
<tr>
<td>(\Theta_y)</td>
<td>(\frac{\rho v_y \Phi}{\mu})</td>
</tr>
<tr>
<td>(\Theta_{x,\phi})</td>
<td>(2 \frac{\rho \Phi}{\mu})</td>
</tr>
<tr>
<td>(\Theta_{y,\phi})</td>
<td>(\frac{\rho v_y}{\mu})</td>
</tr>
</tbody>
</table>

**Y-Momentum transport**

<table>
<thead>
<tr>
<th>variable</th>
<th>definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Phi)</td>
<td>(v_y)</td>
</tr>
<tr>
<td>(C(\Phi))</td>
<td>(v_y)</td>
</tr>
<tr>
<td>(D)</td>
<td>(\mu)</td>
</tr>
<tr>
<td>(D'_{\xi})</td>
<td>0</td>
</tr>
<tr>
<td>(S)</td>
<td>(-P_{iy} + \rho a_{iy} (1 - \beta (T - T_{ref})) - c_p (1 - f_{\xi}) \Phi)</td>
</tr>
<tr>
<td>(\theta)</td>
<td>(\frac{\rho \Phi}{(\Delta t \mu)} - \frac{\rho \Phi}{\Delta t \mu} - S/\mu)</td>
</tr>
<tr>
<td>(\theta_{,\phi})</td>
<td>(\frac{\rho}{(\Delta t \mu)} + \frac{c_p (1 - f_{\xi})}{\mu})</td>
</tr>
<tr>
<td>(\Theta_x)</td>
<td>(\frac{\rho v_x \Phi}{\mu})</td>
</tr>
<tr>
<td>(\Theta_y)</td>
<td>(\frac{\rho \Phi^2}{\mu})</td>
</tr>
<tr>
<td>(\Theta_{x,\phi})</td>
<td>(\frac{\rho v_x}{\mu})</td>
</tr>
<tr>
<td>(\Theta_{y,\phi})</td>
<td>(2 \frac{\rho \Phi}{\mu})</td>
</tr>
</tbody>
</table>
systems of equations

Pressure Poisson equation

\[
\begin{align*}
\Phi & \quad P \\
C(\Phi) & \quad P \\
D & \quad 1 \\
D'\xi & \quad 0 \\
S' & \quad 0 \\
\theta & \quad 0 \\
\theta,\Phi & \quad 0 \\
\Theta_x & \quad -\varrho v_x/\Delta t + \varrho v_{0,x}/\Delta t - \varrho (v_x v_{x,x} + v_y v_{x,y}) \\
& \quad + \mu (v_{x,xx} + v_{x,yy}) + g a_x (1 - \beta (T - T_{ref})) - c P (1 - f_L) v_x \\
\Theta_y & \quad -\varrho v_y/\Delta t + \varrho v_{0,y}/\Delta t - \varrho (v_y v_{y,y} + v_x v_{y,x}) \\
& \quad + \mu (v_{y,xx} + v_{y,yy}) + g a_y (1 - \beta (T - T_{ref})) - c P (1 - f_L) v_y \\
\Theta_x,\Phi & \quad 0 \\
\Theta_{y,\Phi} & \quad 0 \\
\end{align*}
\]

Pressure Poisson equation Neumann boundary conditions

\[
\frac{\partial P}{\partial n} = \left[ -\frac{\partial}{\partial t} (\varrho v) - \nabla \cdot (\varrho v v) + \nabla \cdot (\mu \nabla v) + f \right] \cdot n
\]

Pressure correction Poisson equation

\[
\begin{align*}
\Phi & \quad P' \\
C(\Phi) & \quad P' \\
D & \quad C \\
D'\xi & \quad 0 \\
S' & \quad 0 \\
\theta & \quad 0 \\
\theta,\Phi & \quad 0 \\
\Theta_x & \quad -\varrho v_x \\
\Theta_y & \quad -\varrho v_y \\
\Theta_{x,\Phi} & \quad 0 \\
\Theta_{y,\Phi} & \quad 0 \\
\end{align*}
\]

Pressure correction Poisson equation Neumann boundary conditions

\[
\frac{\partial P'}{\partial n} = 0
\]