Fatigue crack propagation under mixed mode I and III loading

K. Tanaka, Y. Akiniwa
Department of Mechanical Engineering, Nagoya University, Furo-cho, Chikusa-ku, Nagoya 464-01, Japan

Abstract

The propagation behavior of a circumferential crack in a cylindrical steel bar was studied under various combinations of cyclic torsional and axial loadings. The threshold condition of the onset of fatigue crack propagation from a pre-crack was expressed in terms of the stress intensity ranges for mode I and III. The J integral was applied to fatigue crack propagation with excessive plasticity under cyclic loading of mixed mode of I and III.

1 Introduction

Fatigue cracks often show the mixed-mode I and III propagation in power train shafts subjected to reversed torsional and axial loading simultaneously. In comparison with the case of mode I, not many studies have been conducted on mode III or mixed mode of I and III. For damage tolerance design, the direction as well as the rate of crack propagation should be predicted from the loading condition and material inhomogeneities.

In the present paper, the propagation behavior of a circumferential crack in a cylindrical bar of a carbon steel was studied under various combinations of cyclic torque and cyclic axial loading. First, the threshold condition of fatigue crack propagation was discussed in terms of the stress intensity factor (SIF). Next, the J integral was applied to fatigue crack propagation with excessive plasticity under cyclic loading of mixed mode of I and III.
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2 Threshold condition for fatigue crack propagation under mixed mode of I and III

2.1 Models for crack propagation threshold

The propagation path of cracks can deviate from the original crack plane under mode II or III loading. Figure 1 illustrates the formulation of a tilt crack facet and a twist crack facet from the original plane crack. Tilt facets can be formed under mode II loading with or without mode I loading; twist facets under mode III loading with or without mode I loading.

Various criteria proposed for brittle fracture under monotonic mixed-mode loading have been applied to the threshold condition of fatigue crack propagation. They are briefly described as follows with a special reference to the mixed-mode loading of mode I and III.

Erdogan and Sih [1] proposed the maximum hoop stress criterion for predicting the tilt angle of the crack path and the brittle fracture load under mixed-mode I and II.

Pook [2] proposed the maximum principal stress criterion for fatigue crack propagation under mixed mode of I and III. The twist facet angle $\phi$ is the direction of the maximum principal stress, and is given by

$$\phi = \frac{1}{2} \tan^{-1}\left(\frac{2}{1-2\nu} \frac{K_{III}}{K_1}\right)$$

where $K_1$ and $K_{III}$ are SIF for mode I and III, and $\nu$ is Poisson's ratio. The maximum principal stress $\sigma$, is given by

$$\sigma = \frac{K_1(1+2\nu) + \sqrt{K_1^2(1-2\nu)^2 + 4K_{III}^2}}{2 \sqrt{2\pi r}}$$

For the case of mode I, the critical value of $\sigma$ is related to the
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threshold value of SIF, $K_{1th}$, as

$$\sigma_i = \frac{K_{1th}}{\sqrt{2\pi r}}$$

(3)

From Eqs. (2) and (3), we have

$$K_{1th} = \frac{1}{2} \left[ K_1 (1+2\nu) + \sqrt{K_1^2 (1-2\nu)^2 + 4K_{III}^2} \right]$$

(4)

Yates et al. [3] proposed that the tilt angle $\phi$ was determined by the maximum principal stress of the net-section stress

$$\phi = \frac{1}{2} \tan^{-1} \left( \frac{2\tau_N}{\sigma_N} \right)$$

(5)

where $\tau_N$ and $\sigma_N$ are the net-section shear and normal stress. The threshold condition is given by the crack-tip opening displacement perpendicular to the facet plane as

$$\left( \frac{K_1}{K_{1th}} \right)^2 \cos \phi + \sqrt{3(1+\nu)} \left( \frac{K_{III}}{K_{1th}} \right)^2 \sin \phi = 1$$

(6)

where $K_{1th}$ is the critical value of SIF for mode I loading.

The maximum energy release rate criterion is a logical extension of Griffith theory. By denoting the SIF value of twist crack facet by $k_1(\phi)$ and $k_{III}(\phi)$, the energy release rate for crack extension is given by [4]

$$G(\phi) = k_1^2(\phi) \frac{1-\nu^2}{E} + k_{III}^2(\phi) \frac{1+\nu}{E}$$

(7)

The $k_1$ and $k_{III}$ values can be approximated by

$$k_1(\phi) = K_1 \left( \cos^2 \phi + 2\nu \sin^2 \phi \right) + K_{III} \sin 2\phi$$

$$k_{III}(\phi) = K_1 \frac{2\nu-1}{2} \sin 2\phi + K_{III} \cos 2\phi$$

(8)

(9)

The direction of twist crack is given as the direction of the maximum value of $G(\phi)$, and $G_{max}$ is constant at the onset of crack extension.

Local symmetry criterion, or mode II SIF for a tilt crack facet $k_{II} = 0$, is proposed for determining the tilt angle for mixed mode of I and II[5]. The $k_{III} = 0$ criterion for the case of mixed mode of I and III yields the crack direction given by Eq. (1), as far as the approximations of Eqs. (8) and (9) are valid.

In the experiment described in the next section, a cylindrical bar with
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a circumferential pre-crack is subjected under the combination of axial and torsional loadings. The radius of the bar is 16 mm and the depth of the pre-crack is 1.9 mm. For this geometry, the twist angle calculated from Eqs. (1) and (5), and also from $G_{\max}$ model is shown Fig. 2, where the mode mixity parameter is defined by

$$\beta = \tan^{-1} \left( \frac{K_{III}}{K_1} \right)$$  \hspace{1cm} (10)

and $\nu$ is 0.279. The threshold condition derived from the three models is

Figure 2. Facet angle predicted under mixed mode of I and III.

Figure 3. Crack propagation threshold under mixed mode of I and III.
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shown in Fig. 3. $G_{\text{max}}$ model gives the same prediction as Pook's model as far as $K_{\text{III}}/K_{\text{I}}$ is less than 2.05 or $\beta$ is less than 64.0 deg. The threshold of mode $\text{III}$, $K_{\text{III}}$, is equal to $K_{\text{Ith}}$ for Pook's model, $K_{\text{III}}=0.79 K_{\text{th}}$ for Yates' model, and $K_{\text{III}}=0.85 K_{\text{th}}$ for $G_{\text{max}}$ model. The twist angle for mode $\text{III}$ is 45 deg for Pook's and Yates' model, and 0 deg for $G_{\text{max}}$ model.

2.2 Experimental procedure

A cylindrical bar of carbon steel (JIS S45C) with the diameter of 16 mm was circumferentially notched, and annealed at 850°C for 1 hr. The yield strength was 311 MPa, and the tensile strength was 588 MPa. The Young's modulus was 216 GPa and Poisson's ratio was 0.279.

A pre-crack of a depth of 0.40 mm was introduced below a circumferential notch (with a depth of 1.5 mm) by cyclic compression. All the specimens were annealed at 650°C for 1 hr before fatigue tests.

The fatigue tests were conducted in a computer-controlled electro-servo hydraulic tension-torsion fatigue testing. Figure 4 shows the load wave applied. The load ratio of the torque $T$ is $R=-1$; that of the axial load $P$ is $R=0$. The frequency is 10 Hz for the torque and 20 Hz for the axial load. The mode ratio $\Delta K_{\text{III}}/\Delta K_{\text{I}}$ adopted for tests was 0, 0.4, 1.0, 2.4 and $\infty$, where $\Delta K_{\text{III}}$ and $\Delta K_{\text{I}}$ mean the total range of SIF including reverse loading.

The direction of crack extension was conducted by the d.c. electrical potential method [6]. The change of the potential deference of 1 $\mu$V corresponded to the crack extension of about 15 $\mu$m in our experimental system. The specimens in which the crack extension had been detected by the d.c. potential method were fractured at liquid nitrogen temperature. The fracture surface was examined with a scanning electron microscope (SEM).

![Figure 4. Loading waves in mixed mode (I +III) loading.](image)
2.3 Experimental results and discussion

The experimental results of the threshold tests were shown in Fig. 5, where the open circles indicate no-growth and the solid circles indicate crack growth detected. The threshold range of SIF for mode I is $\Delta K_{I_{th}} = 3.3\text{MPa}\sqrt{\text{m}}$, and that for mode III is $\Delta K_{III_{th}} = 3.0\text{MPa}\sqrt{\text{m}}$. The experimental data can be approximated by a quarter ellipse expressed by

$$\left(\frac{\Delta K_I}{\Delta K_{I_{th}}}\right)^2 + \left(\frac{\Delta K_{III}}{\Delta K_{III_{th}}}\right)^2 = 1$$ \hfill (11)

![Figure 5. Threshold conditions for fatigue crack propagation under mixed mode (I + III) loading](image)

(a) Positive torque at point A  \hspace{1cm} (b) Negative torque at point B

![Figure 6. Stress state at crack tip under mode I and III loading.](image)
For the case of torsional loading with $R = -1$, the principal stress direction is different between positive torque and negative torque. Figure 6 illustrates the switching of the principal stress direction due to the change of the direction of the applied torque. To apply Pook's and Yates' model [2,3] to the wave form (Fig. 4) of the present experiment, it is unknown how to deal with the switching of the principal stress direction.

By overlooking the above switching, $\Delta K_1$ and $\Delta K_iii$ are substituted for $K_1$ and $K_iii$ in Eqs. (1) and (4) for Pook's model, and in Eqs. (5) and (6) for Yates' model. The relations are shown in Fig. 7, where $\Delta K_1$ and $\Delta K_iii$ are normalized by $\Delta K_{1_{th}}$. The experimental data seem to fit the prediction by Yates' model.

Figure 8 shows the SEM micrographs of the fracture surface of the specimen subjected to the cyclic load just above the threshold, where A, B, and C indicate the pre-crack face, the fatigue crack face and the brittle fracture surface, respectively. The face of fatigue crack propagation is inclined to the pre-crack face in Fig. 8(a) with $\Delta K_iii = 3.0$ MPa$\sqrt{m}$, while that is parallel to the pre-crack face in Fig. 8(b) with $\Delta K_iii = \Delta K_1 = 2.3$ MPa$\sqrt{m}$. According to the model prediction shown in Fig. 2, the tilt angle is around 30 to 40 deg. for the case of $\Delta K_iii/\Delta K_1 = 1$. The prediction does not agree with the experimental results. Further developments of modelling are necessary to include the effect of various loading conditions.

![Figure 7. Experimental results compared with model prediction.](image-url)
Figure 8. Scanning electron micrographs of fracture surface of specimens just above the threshold.

(a) $\Delta K_i = 3.0 \text{MPa}\sqrt{\text{m}}$, $R = -1$.

(b) $\Delta K_i / \Delta K_t = 1.0$, $\Delta K_i = 2.3 \text{MPa}\sqrt{\text{m}}$, $\Delta K_t = 2.3 \text{MPa}\sqrt{\text{m}}$. 
3. J integral approach to fatigue crack propagation with excessive plasticity under mixed mode of I and III

3.1 J integral approach

A circumferential crack in cylindrical bars shows coplanar mode III propagation under high cyclic torsional loading. Because of excessive plasticity, linear elastic fracture mechanics is no longer applicable to this case. J integral approach was first applied to mode III propagation by Tanaka et al. [6]. They estimated the J integral range from the loading part of the hysteresis loop of torque versus angle of twist.

Figure 9 illustrates the hysteresis loops of torque versus angle of twist and of load versus displacement obtained under mixed loading of mode I

![Figure 9. Evaluation of J integral range.](image-url)
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and III. The point of crack-tip opening was determined from the unloading compliance method. The mode III J integral range, $\Delta J_{III}$, is estimated from the loading part above the opening point by

$$\Delta J_{III} = \frac{\Delta K_{III}}{E} (1 + \nu) + \frac{3}{2\pi b^2} U_p$$

where $b$ is the radius of the ligament of the specimen, $\Delta K_{III}$ is the mode III effective stress intensity range and $U_p$ is the energy corresponding to the shaded area shown in Fig. 9(a). The mode I J integral is estimated by

$$\Delta J_1 = \frac{\Delta K_{I}}{E} (1 - \nu^2) + \frac{1}{2\pi b^2} \left[ 3U - \Delta P_{eff} \cdot \Delta u - \frac{1}{2} \Delta P_{eff} \cdot \Delta u_e \right]$$

where $\Delta K_{I}$ is the mode I effective stress intensity range, $U$ corresponds to the shaded area in Fig. 9(b), and $\Delta P_{eff}$, $\Delta u$, $\Delta u_e$ are also shown in Fig. 9(b).

For the case of mixed mode loading, the total range $\Delta J$ can be given as the sum of $\Delta J_1$ and $\Delta J_{III}$:

$$\Delta J = \Delta J_1 + \Delta J_{III}$$

3.2 Experimental procedure

The material and the specimen were the same as those described in Chapter 2. The notch depth of the specimens used for mode III experiment was 1.5 and 3.5 mm; that for mixed mode and mode I experiments was 3.0 mm. All the specimens were pre-cracked by a depth of 0.40 mm under cyclic compression, and then annealed at 650 °C for 1 hr before fatigue tests.

The fatigue crack propagation test was conducted with $R = -1$ both for torque and axial load under in-phase condition. The range of J integral, $\Delta J$, was kept constant for mode III crack propagation tests. For the other tests, the torsional angle and axial displacement were controlled. The crack length was measured by the d.c. potential method.

3.3 Experimental results and discussion

The change of crack propagation rate with the crack length for the mode III case is shown in Fig. 10, where the notch depth $t$ is 1.5 mm and $c$ is the crack length from the pre-crack. The crack propagation rate decreases with crack extension even under a constant $\Delta J_{III}$ condition. This decrease is caused by the sliding contact of crack faces as reported by Tschegg [7]. The crack propagation rate without contact shielding is obtained by extrapolating the dc/dN vs. $c$ relation to the zero crack length as shown in
Figure 10. Change of crack propagation rate with crack length.

Figure 11. Relation between crack propagation rate and $\Delta J$. 
Fig. 10. The crack propagation rate thus determined is plotted against $\Delta J_{\text{III}}$ in Fig. 11. The relation between $dc/dN$ (m/cycle) and $\Delta J$ (N/m) is approximated by the following power equation:

$$\frac{dc}{dN} = C_{\text{III}} \Delta J_{\text{III}}^{m_{\text{III}}}$$  \hspace{1cm} (15)

where $C_{\text{III}} = 7.03 \times 10^{-13}$ and $m_{\text{III}} = 1.43$. This relation is rather independent of the initial notch depth.

For the case of mode I loading, the crack-opening point was determined from the load vs. displacement curve, and $\Delta J_{\text{I}}$ was estimated by using Eq. (13). The relation between $dc/dN$ (m/cycle) and $\Delta J_{\text{I}}$ (N/m) was shown with the solid circles in Fig. 11. The relation is expressed by

$$\frac{dc}{dN} = C_{\text{I}} \Delta J_{\text{I}}^{m_{\text{I}}}$$  \hspace{1cm} (16)

where $C_{\text{I}} = 1.39 \times 10^{-13}$ and $m_{\text{I}} = 1.71$.

In displacement-controlled tests of mixed mode loading, the $J$ integral ranges of mode I, $\Delta J_{\text{I}}$, and of mode III, $\Delta J_{\text{III}}$, were nearly constant during the crack extension of 1 mm from the pre-crack. The ratio of $\Delta J_{\text{III}} / \Delta J_{\text{I}}$ tested was 0.5, 1, and 2. During this crack extension, the crack propagation rate was also nearly constant or decreased by a small amount. In Fig. 11, the mean propagation rate is plotted against the total range of $\Delta J$. All the data lie between the relations for mode I and mode III.

Figure 12 shows the change of the crack propagation rate at $\Delta J = 2 \times 10^4$ N/m with the mode mixity parameter defined by

$$\beta^* = \tan^{-1} \left( \frac{\Delta J_{\text{III}}}{\Delta J_{\text{I}}} \right) \text{ deg}$$

Figure 12. Change of crack propagation rate with mode mixity.
(a) $\Delta J = 2.6 \times 10^4 \text{ N/m, } \Delta J_4 / \Delta J_1 \approx 0, \text{dc/dN} = 1.6 \times 10^4 \text{ m/cycle.}$

(b) $\Delta J = 2.1 \times 10^4 \text{ N/m, } \Delta J_4 / \Delta J_1 \approx 1, \text{dc/dN} = 1.8 \times 10^4 \text{ m/cycle.}$

Figure 13. Scanning electron micrographs of fatigue fracture surfaces.
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\[ \beta^* = \tan^{-1} \left( \frac{\Delta J_{\text{III}}}{\Delta J_{\text{I}}} \right) \]  

(17)

The bar attached to the data point indicates the range of variation of the crack propagation rate. The crack propagation rate decreases with increasing ratio of \( \Delta J_{\text{III}}/\Delta J_{\text{I}} \). The line in the figure is drawn under the assumption that the crack propagation rate is given as the sum of the rates by mode I and III loadings as

\[ \frac{dc}{dN} = C_1 \Delta J_{\text{I}}^{m_1} + C_{\text{III}} \Delta J_{\text{III}}^{m_{\text{III}}} \]  

(18)

Equation (18) seems to express the crack propagation rate under mixed mode loading.

The fatigue fracture surface of all the specimens was macroscopically flat and did not show factory-roof type topography. SEM micrographs of fatigue fracture surface are shown in Fig. 13, where (a) is for mode III (\( \Delta J_{\text{III}} = 2.6 \times 10^4 \text{ N/m}, \frac{dc}{dN} = 1.6 \times 10^{-5} \text{ m/cycle} \)) and (b) is for a mixed mode (\( \Delta J = 2.1 \times 10^4 \text{ N/m}, \frac{\Delta J_{\text{III}}}{\Delta J_{\text{I}}} = 1, \frac{dc}{dN} = 1.8 \times 10^{-5} \text{ m/cycle} \)). The crack propagation direction is from top to bottom. On the fracture surface made under mode III, there are no striations. Only rub marks are seen extending perpendicular to the growth direction. On the other hand, striations can be observed on the fracture surface made under mixed mode loading. Those striations suggest that the mechanism of mode I crack propagation is operating in mixed-mode propagation.

4 Conclusions

By using a circumferentially cracked steel bar subjected to various combinations of cyclic torque and axial loading, the effect of the mode mixity of I and III on the threshold condition of the onset of crack propagation and on the propagation rate of fatigue cracks with excessive plasticity was studied. The followings are the summary of the results obtained:

(1) The threshold condition of fatigue crack extension was expressed by

\[ \left( \frac{\Delta K_{\text{I}}}{\Delta K_{\text{I,th}}} \right)^2 + \left( \frac{\Delta K_{\text{III}}}{\Delta K_{\text{III,th}}} \right)^2 = 1 \]

where \( \Delta K_{\text{I,th}} \) and \( \Delta K_{\text{III,th}} \) are the threshold values of the stress intensity ranges for mode I and III.

(2) The models proposed by Pook and Yates for the stress ratio \( R \geq 0 \) give conservative estimates for the present experiments for the threshold under \( R = -1 \).

(3) The angle of the fracture facet made at the tip of pre-cracks just above the threshold did not agree with the maximum principal stress direction. Further refinement of modelling of the threshold is necessary.
(4) Under high cyclic torque, mode III cracks propagated macroscopically coplanar to the pre-crack plane. The crack propagation rate $dc/dN$ (m/cycle) was expressed as a function of the J integral range $\Delta J$ as

$$
dc / dN = C_{\text{III}} \Delta J_{\text{III}}^{m_{\text{III}}}
$$

where $C_{\text{III}} = 7.03 \times 10^{-13}$ and $m_{\text{III}} = 1.43$.

(5) For the case of mode I loading, the relation obtained was

$$
dc / dN = C_{I} \Delta J_{I}^{m_{I}}
$$

where $C_{I} = 1.39 \times 10^{-13}$ and $m_{I} = 1.71$. The propagation rate is lower under mode III loading than under mode I loading, when compared at the same $\Delta J$ value.

(6) The data of the crack propagation rate under mixed-mode loading as a function of $\Delta J$ lie between the relations for mode I and III.

(7) The crack propagation rate under mixed mode loading can be expressed as

$$
dc / dN = C_{I} \Delta J_{I}^{m_{I}} + C_{\text{III}} \Delta J_{\text{III}}^{m_{\text{III}}}
$$

References