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Boundary element formulations in fracture mechanics: a review

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1 Introduction

The modern Boundary Element Method (BEM) originated from work carried out by a few research groups in the 1960's on the application of boundary integral equations for the solution of engineering problems. These researchers were seeking a different solution from the Finite Element Method (FEM) which was starting to become more widely established for computational analysis of engineering problems.

Boundary integral methods in structural analysis were known in the western countries through the work of Russian authors such as Muskelishvili, Mikhlin and Kupradze. These methods at that time were considered to be difficult to implement numerically.

The "direct" boundary element formulation can be traced back to Kupradze. Early work by Jaswon[1] provided the foundation for subsequent direct formulation in engineering. Later Rizzo[2] presented the direct formulation for elastostatic problems by the application of Betti's and Somiglina's formulae. During the same period Shaw[3] and Cruse[4] presented an indirect and direct formulations respectively for elastodynamic problems.

During the 1960's a small group at Southampton University started working on the application of integral equations to solve stress analysis problems. The work was continued through a series dealing mainly with elastostatic problems under supervision of Carlos Brebbia. Lachat's work in Brebbia's group was the first contribution of the use of higher order elements for elastostatics[14]. This capability marked an important development, as until then, integral equations were restricted to constant sources and were assumed to be concentrated as a series of points on the external surface of the body. The constant source approach gave poor results in many practical applications and in particular those involving bending.

In 1977 Jaswon and Symm[5] published a book on integral equation methods. Their book, which contained considerable original material also illustrated the equivalence between Rizzo's elastostatic formulation and Kupradze's.

In 1978 the first book with Boundary Elements as its title, written by Brebbia, was published [6]. The importance of this book is that it pointed out the relationship between the BEM an other methods such as FEM. Brebbia was also the first to present a derivation of the boundary integral equation from a weighted residual formulation. More mathematical aspects of the method were presented by Brebbia and Walker[7]. Later, in early 80's as the BEM was rapidly advancing there was a need to define the state of the art on the subject and a more comprehensive and definitive book was written by Brebbia, Telles and Wrobel[8].

This paper reviews advances in the application of the boundary element method (BEM) to fracture mechanics that have taken place over the last 25 years. Over this period the method has emerged as the most efficient technique for the evaluation of stress intensity factors (SIF) and crack growth analysis in the context of linear elastic fracture mechanics (LEFM). Much has also been achieved in the application to dynamic fracture mechanics.

This paper reviews the modelling strategies that have been developed as well as applications to: LEFM, SIF calculations, dynamics, anisotropic and composite materials, interface cracks, non-metallic materials, thermoelastic problems, non-linear problems and crack identification techniques. Special attention has also been given to indirect boundary integral equation formulations.

2 Crack Modelling Strategies

Straightforward applications of the boundary element method to crack problems leads to a mathematical degeneration, if the two crack surfaces are considered co-planar, as was shown by Cruse[9]. For symmetrical crack geometries, it is possible to overcome this difficulty by imposing the symmetry boundary condition and hence modelling only one crack surface. However, for non-symmetrical crack problems, another way must be found. Curse and Van Buren[10] explored the possibility of modelling the crack as a rounded notch with an elliptical closure, but this model required many elements to model the tip of the rounded notch. The reported accuracy for the stress intensity factor of the centre-crack-tension-specimen was poor, with errors of around 14%.

Snyder and Cruse[11] introduced a special form of fundamental solution for crack problems in anisotropic media. The fundamental solution (Green's function) contained the exact form of the traction free crack in an infinite medium, hence no modelling of the crack surfaces was required. The crack Green's function technique although accurate, is limited to two-dimensional straight cracks. For kinked cracks, the region must be divided into segments with straight cracks see Kuhn[12]. However, this approach is inefficient as it introduces additional elements into the model. The first widely applicable method for dealing with two co-planar crack surfaces was devised by Blandford et. al[13]. This approach which is based on a multi-domain formulation is general and can be applied to both symmetrical and anti-symmetrical crack problems in both two- and three-dimensional configurations. The multi-region method introduces artificial boundaries into the body, which connect the cracks to the boundary, in such a way that each region contains a crack surface. The two regions are then joined together such that equilibrium of tractions and compatibility of displacements are enforced. The main drawback of this method is that the introduction of artificial boundaries are not unique, and thus cannot be implemented into an automatic procedure. In addition, the method generates a larger system of algebraic equations than is strictly required. Despite these drawbacks, the subregion method has been widely used for crack problems.

More recently the Dual Boundary Element Method (DBEM) as developed by Portela, Aliabadi and Rooke[15] for two-dimensional problems and Mi and Aliabadi[16] for three-dimensional problems has been shown to be, a general and computationally efficient way of modelling crack problems in BEM. General mixed-mode crack problems can be solved with DBEM, in a single region formulation, when the displacement boundary integral equation is applied on one of the crack surfaces and the traction boundary integral equation on the other. In the context of the direct BEM, the dual equations were first presented by Watson[17], in a formulation based on the displacement equation and its normal derivative. Dual boundary equations have been applied to solve three-dimensional potential theory by Gray and Giles[18], Rudolphi et. al. [19] and in three-dimensional elastostatics by Gray et. al.[20].

The main difficulty in the DBEM formulation is the development of a general and accurate modelling procedure for the integration of Cauchy and Hadamard principal value integrals appearing in the traction equation. The necessary conditions for the existence of these singular integrals, assumed in the derivation of the dual boundary integral equa-

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tions, imposes certain restrictions on the choice of basis functions for the crack surfaces. In the point collocation method of solution, the displacement integral equation requires the continuity of the displacement components at the nodes (i.e. collocation points), and the traction integral equation requires the continuity of the displacement derivatives at the nodes. These requirements were satisfied in [17] by adopting the Hermitain elements, however, the solutions reported were not very accurate. Recently Watson[21] has improved the accuracy of this formulation. Rudolphi et. al.[19] reported unexplained oscillations in their results, while Gray et. al.[20] devised a scheme based on a special integration path around the singular point for linear triangular elements. The formulation in [19,20] were applied to embedded cracks only. In [15,16] both crack surfaces were discretized with discontinuous quadratic elements; this strategy not only automatically satisfies the necessary conditions for the existence of the Hadamard integrals, but also circumvents the problem of collocating at crack kinks and crack-edge corners. Several examples including embedded, edge, kinked and curved cracks were solved accurately in [15,16]. For other contributions in DBEM see for example Lutz[22] and Hong and Chen[23]

Detailed description of some of the advanced BEM formulations can be found in Aliabadi and Brebbia[24].

3 Linear Elastic Fracture Mechanics

The application of the BEM to Linear Elastic Fracture Mechanics (LEFM) is now well established and widely used in practice. The method offers a clear advantage over other methods such as the Finite Element Method for LEFM. One of the main reasons for this advantage is the possibility of evaluating the Stress Intensity Factors (SIF) accurately. There have been many methods devised for the evaluation of SIF's using BEM. The most popular are perhaps the techniques based on the quarter-point elements, path independent contour integrals, energy methods, subtraction of singularity method and the weight function methods. A detailed description of these methods can be found in the text book by Aliabadi and Rooke[25].

The use of quarter-point elements in three-dimensional boundary element analysis was reported by Cruse and Wilson[26] who also introduced additional modifications for modelling singular tractions. Several ways of evaluating the stress intensity factors from the displacements on the crack surfaces have been proposed by researchers[27],[28],[29]. Smith and Mason[30] demonstrated the use of quarter-point element for curved cracks. Martinez and Dominguez[29] proposed an alternative way of obtaining the SIF's for the quarter-point elements. Their method which relates the so called tractions at the crack tip to the stress intensity factors is more efficient than the displacement based formulae. A comparison of methods of evaluating the SIF's from the quarter-point elements has been reported by Smith[28]. Other special crack tip elements for modelling the near crack tip behaviour are reported by Aliabadi[31], Jia, Shippy and Rizzo[32] for two-dimensional problems and Luchi and Rizzuti[33] for 3D continuous elements and Mi and Aliabadi[34] for 3D discontinuous elements. Zamani and Sun[35] have proposed a hybrid type element. Their proposed element is similar to the enriched element used in the finite element method, where the crack tip stress fields are added to the standard Lagrangian polynomials.

The use of path independent contour integrals has also been popular in BEM, as the stress intensity factors can generally be evaluated by a post-processing procedure. Boissenot, Lachat and Watson[36] reported the use of J-integral for 3D symmetric crack problems. Later, Kishitani, et. al.[37] and Karami and Fenner[38] reported its use for several 2D symmetrical problems. Aliabadi[39] applied the J-integral and BEM to mixed-mode crack problems and decoupled the J into its symmetrical and anti-symmetrical components. It was shown in [39] that accurate values of mode I and mode II stress intensity factors can be obtained from the J-integral. Man, Aliabadi and Rooke[40] utilized the mixed-mode J-integral to study the effect of contact forces on the crack behaviour. The application of the J-integral to mixed mode 3D problems was presented by Rigby and Aliabadi[41] and Huber and Khun[42]. Sollero and Aliabadi[43] proposed an alternative method for decoupling the mixed-mode J-integral based on the crack opening/sliding displacements ratio. Soni and Stern[44] and Stern et. al.[45] developed a path independent integral and used the BEM to evaluate mixed-mode stress intensity factors. More recently, Wen, Aliabadi and Rooke[46] developed an alternative path independent integral for the evaluation of mixed-mode stress intensity factors. In [46] an indirect boundary element formulation was used to evaluated interior values of displacements and stresses. Bainbridge, Aliabadi and Rooke[47] have proposed a path independent integral for 3D problems. Their path independent integral utilizes solutions due to point forces on straight fronted and penny shaped cracks as an auxiliary field. Mixed-mode stress intensity factors can be evaluated with this technique.

Another way of calculating SIF's is from the use of strain energy release rate G. However, this method requires several computer runs for 3D problems. Cruse and Meyers[48] proposed a technique for 3D problems which limited the computer runs to two. The two computer runs consisted of one for the original crack front and one for the perturbed crack front, obtained by moving all the nodes on the crack front radially along lines normal to the crack front. Cruse and Meyers[48] used linear triangular elements. Later Tan and Fenner[49] used quadrilateral elements with quadratic variations to represent both the surface and the unknown functions. Further development of BEM using the strain energy release rate has been reported by Bonnet[50].

The methods discussed above are based on attempts to model the singular behaviour of stresses near the crack tip. In contrast, the subtraction of the singularity method avoids the need for this task; it removes the singular fields completely. This leaves a non-singular field to be modelled numerically. This approach was first introduced in BEM by Papamichel and Symm[51] for analysis of a symmetrical slit in potential problems. Xanthis et. al.[52] used this formulation to solve the same problem of a symmetrical slit using quadratic isoparametric elements. The extension of the method to two-dimensional elasticity was presented by Aliabadi et. al.[53], [54]. who obtained both mode I and mode II stress intensity factors. This formulation was extended to V-notch plates in [55]. The application of the method to 3D problems is reported by Aliabadi and Rooke [56].

Methods for the evaluation of stress intensity factors from the crack Green's functions have been proposed by Mews[57] for kinked cracks and Dowrick[58] and Young et. al.[59] for stiffened panels. Recently Telles et. al.[60] have proposed to evaluate the crack Green's function numerically.

An alternative method to the usual stress analysis for the evaluation of stress intensity factors is the weight function method. The advantage of the weight functions lies in their universality, that is they are independent of the loading. Hence, once the weight functions are evaluated for a given crack geometry, they can be used to evaluate the stress intensity factors for any applied loading. Bueckner introduced the concept of weight functions in the early 70's. His weight functions satisfy the linear equations of elasticity, but have a strong singularity at the crack tip. He refers to them as "fundamental fields". Later, Rice showed that the weight functions could be equally well determined by differentiating known elastic solutions for displacement fields with respect to the crack length. For details of these two formulations readers should consult Aliabadi and Rooke[25].

Cruse and Besuner[61] and Besuner[62] developed a BEM strategy for evaluating the weight functions based on Rices's derivation. In their work, a 3D BEM analysis was used to calculate average stress intensity factors for each perturbation of the crack front. The instantaneous values at a specific point and the average value along the whole crack front are not exactly equivalent for most 3D problems since the stress intensity factors are not generally constant. Further, this technique requires many iterations to obtain a single stress intensity factor solution and is thus computationally expensive. Another technique

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using Rice's derivation is due to Heliot, Labbens and Pellisier-Tannon[63]. This technique is the extension of the approximate polynomial distribution as proposed by Grant (see Aliabadi and Rooke[25]). In this work the polynomial influence functions were defined to correspond to the terms of a polynomial expansion of the stress fields acting on the crack faces; these influence functions also depended on the radii and depth of the semi-elliptical crack. Numerical crack-face weight functions were obtained after five computer runs, one for each term in the polynomial. Later Cruse and Ravendera[64] developed a two-dimensional BEM procedure based on the Rice's formulation. In their work, the crack Green's function was utilized. Accurate values of stress intensity factors were reported for symmetrical crack problems. Recently, Wen, Aliabadi and Rooke[65],[66] have developed a BEM technique for evaluating 2D and 3D weight functions. They used a displacement discontinuity method and fictitious stress method to obtain weight functions for mixed-mode problems according to Rice's derivation.

Cartwright and Rooke[67] showed that a boundary element analysis produced stress intensity factors which are more accurate and efficient than the equivalent finite element analysis. This formulation which is based on Bueckner's fundamental fields, has been extended by Aliabadi, Cartwright and Rooke[68] to both mode I and mode II deformations which, in this formulation are independent. The improvement to this model was reported by Aliabadi, Rooke and Cartwright[69] for two-dimensional problems by employing the subtraction of singularity technique. Bains, Aliabadi and Rooke[70],[71] presented a boundary element method for evaluating 3D weight functions based on the subtraction of singular fields. They derived and utilized fundamental fields for straight fronted and penny shaped cracks. The application of this method was demonstrated for a wide range of crack problems.

4 Cracks in Anisotropic and Composite Materials

One of the first application of BEM to cracks in anisotropic materials was due to Snyder and Cruse[11]. In this work the crack Green's function was used as a procedure for embedding an exact crack modelling in the boundary integral representation. This approach proved popular with several authors for example Konish[72], Chan and Cruse[73], Kamel and Liaw[74] and Liaw and Kamel[75]. However, as stated earlier this approach is limited in its application. The multi-region method and quarter-points have been used by Tan and Gao[76] to solve several crack problems in orthotropic materials. Sollero and Aliabadi[77] presented a multi-region method together with a mixed-mode J-integral for crack problems in orthotropic and anisotropic materials. Doblare, Espiga and Alcantud[78] have also used the multi-region BEM formulation. Ishikawa[79] and Sladek and Sladek[80] presented BEM results for 3D crack problems in anisotropic materials. More recently Sollero and Aliabadi[81] presented a dual boundary element formulation for cracks in anisotropic materials. They utilised a J-integral formulation to obtain accurate stress intensity factors for several mixed-mode problems.

The application of BEM to cracking in composite materials has been reported by Shilko and Shcherbakov[82], Tan and Bigelow[83], Kamel et al.[84] and Klingbeil[85]. More recently, Bush[86] analysed the fracture of particle reinforced composite materials with BEM. Nonlinear behaviour of metal matrix fiber composites with damage on the interface has been analysed by Shibuya and Wang[87]. Shan and Chou[88] have analysed the problem of fiber/matrix interfacial debounding. Chella, Aithal and Chandra[89] studied a quasi-static crack extension in fiber-reinforced composites subjected to thermal shock. Sensitivity analysis based on the adjoint formulation was developed in [89] to evaluate the energy integrals in cracked bodies.

5 Interface Cracks

The use of Hetenyi's fundamental solution in BEM to avoid modelling the interface of two different materials was introduced by Yuuki et al[90] and Yuuki and Cho[91]. In these papers several interface crack problems were also analysed. The use of the multi-region method for interface cracks has been reported by Lee and Choi[92] and Tan and Gao[93]. Other BEM solutions to interface crack problems are reported by Battachayya and Willment[94], Kwon and Dutton[95]. Tan and Gao[96] developed quarter-point elements to model interface cracks between dissimilar materials in axisymmetry. Special procedures were developed to deal with the oscillatory singular nature of the stresses. A three-dimensional BEM for analyses of interface cracks and dissimilar material joints has been presented by Yuuki and Xu[97]. The application of the virtual crack extension and a contour integral technique to interfaces cracks have been presented by Miyazaki et. al.[98],[99]

6 Dynamic Fracture Mechanics

The boundary element solutions to elastodynamic problems are usually obtained (see Manolis & Beskos[100]; Dominguez[101]; Brebbia and Nardini[102]) by either the time domain, Laplace or Fourier transforms or the dual reciprocity method.

Nishimura, Guo and Kobayashi[103],[104] used the time domain method to solve crack problems. They used the double layer potential formulation which contains the hyper-singular integrals. The equations were regularized by using integration by parts twice. The constant and linear shape functions were used for the spatial and temporal approximations respectively. The method was applied for stationary and growing straight cracks in 2D, and plane crack in 3D infinite domains. The dynamic stress intensity factors were obtained for the crack opening displacements (CODs). Later Zhang and Achenbach[105] improved the crack modelling used in [104] by utilizing constant elements away from the crack front and spatial square-root functions near the crack tip. They analysed collinear cracks in an infinite domain. Hirose[106] and Hirose and Achenbach[107], [108] applied the formulation based on the traction equation with picewise linear temporal functions to both constant and growing penny-shaped cracks. Zhang and Gross[109] used the two-state conservation integral of elastodynamic, which leads to non-hypersingular traction integral equations. The unknowns in this approach are the crack opening displacements and their derivatives. This formulation was applied to penny shaped and square cracks in infinite domains.

Nicholson and Mettu[110] and Mettu and Nicholson[111] used two types of approximation: i) constant elements for both spatial and temporal interpolation of boundary quantities; and ii) quadratic in space and linear in time. The method was applied to solve several opening-mode crack problems. Dominguez and Gallego[112] used a mixed variation of boundary values in which tractions were assumed to be constant and displacements linear in time. The boundaries were divided into quadratic elements. At the crack tips ordinary and traction-singular quarter-point elements (QPEs) were used. The dynamic stress intensity factors were determined using the COD and tractions of traction-singular elements. The method was applied to finite bodies with cracks. Mixed-mode crack problems were analysed using the subregion technique.

Siebrits and Crouch[113] have presented a time-domain displacement discontinuity formulation for 2D problems. In their formulation linear, continuous in time and piece-wise linear in space interpolation functions were assumed for the displacement discontinuities.

The dual boundary element formulation in the time domain was presented by Fedelinski, Aliabadi and Rooke[114]. The temporal variation of the boundary displacements and tractions was approximated by piecewise linear and constant functions, respectively. The dynamic stress intensity factors were calculated using the crack opening displacements and the path independent $\hat{J}$-integral. This method was used to study dynamic behaviour of stationary cracks in finite and infinite domains in two-dimensional analysis. Both mode I and mixed-mode crack problems were considered.

An application of a Laplace transform method was presented by Sladek and Sladek[115],

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who analysed a penny-shaped crack in an infinite elastic domain subjected to harmonic and impact loads on crack surfaces. The problem was solved using the traction integral equation in terms of the displacement discontinuity. A similar method was used by Yin and Li[116] to analyse a rectangular plate with a central crack. The influence of different lengths of the crack and different time-dependent loadings were studied. The dynamic stress intensity factors were calculated from the COD of quarter-point elements and the extrapolation technique. Tanaka et. al.[117] used a similar method to calculate dynamic stress intensity factors. The effect of varying the number of boundary elements and the number of parameters in the Durbin and Hosono method of inverting was investigated. Polyzos et. al.[118] analysed rectangular plate of viscoelastic material with either a central or an edge inclined crack. The mixed-mode crack problem was solved using the multi-region method.

Wen, Aliabadi and Rooke[119],[120] presented a Laplace transform displacement discontinuity and fictitious stress formulations for two- and three-dimensional problems. They obtained the stress intensity factors for many 2D and 3D problems using an equivalent stress approach.

Fedelinski, Aliabadi and Rooke[121],[122] presented a Laplace transform dual boundary element formulation for two-dimensional problems. Several mode I and mixed-mode crack problems were solved and the stress intensity factors were evaluated using the quarter-point elements.

Wen, Aliabadi and Rooke[123],[124] presented a formulation for the evaluation of dynamic weight functions in two- and three-dimensions. These weight functions are independent of both spatial distribution and time variation of the loading. The advantage of these features were demonstrated for several mixed-mode problems.

The Fourier transform method was used by Chirino and Dominguez[125] to analyse cracks in an infinite plane, a half-plane and a finite domain. Hirose[126] used the traction equation and the displacement discontinuity method to investigate the scattering of elastic waves from a penny shaped crack.

The dual reciprocity method was used by Balas, Sladek and Sladek[127] for symmetric crack problems. Pekau and Batta[128] used the subregion method for stationary and growing cracks in a rectangular plate. A similar method was used by Chirino, Gallego, Saez and Dominguez[129], who also used the subregion technique and presented a comparative study of different approaches.

Fedelinski, Aliabadi and Rooke[130],[131] presented a DBEM formulation for dynamics using the dual reciprocity method. The stress intensity factors for mixed-mode problems were obtained using COD and the J-integral method.

A comparison of the time-domain, Laplace transform and dual reciprocity method in terms of computing time and storage as well as accuracy have been presented by Fedelinski, Aliabadi and Rooke[132].

7 Thermoelastic Fracture Mechanics

One of the early application of BEM to thermoelastic crack problem is due to Kuhn[133] and Predeleanu and Screpel-Fleurier[134]. Later Tanaka, Togoh and Kikuta[135] used a BEM formulation with a domain term to represent the temperature field. Lee and Cho[136] solved several symmetrical crack problems. Sladek and Sladek[137] solved transient symmetrical crack problems. Tanaka et. al.[138] applied the multi-region method with domain discretization. Raveendra and Banerjee[139] utilized the multi-region formulation to thermoelastic crack problems using a boundary only formulation. Quarter-point elements were used in [139] and the stress intensity factors were evaluated from the crack opening displacements. Liu and Alterio[140] presented a series of solutions to mode I and mode II crack problems. Recently Prasad, Aliabadi and Rooke[141],[142] have presented a BEM formulation for mixed-mode crack problems in static and transient thermoelasticity. In their formulation two pairs of boundary integral equations were employed. One pair consists of temperature and displacement, and the other pair of flux and traction. The stress intensity factors were evaluated from both the quarter-point elements and a J-integral formulation. A special consideration to the thermal singularity at the crack tip is addressed in [143].

8 Nonlinear Fracture Mechanics

One of the first attempts in applying elastoplastic boundary element formulations to fracture mechanics was made by Morjaria and Mukherjee[144] and Banthia and Mukherjee [145], [146]. In their approach the crack Green's function was used to model the crack. Later Cruse and Polch[147] also used the Green's function approach together with an improved model of the crack. Tan and Lee[148] used the Kelvin's fundamental solution and introduced the crack by specifying appropriate boundary conditions. In their study the behaviour of an internally pressurized thick-walled cylinder containing a radial crack was investigated. Yong and Guo[149] have also studied a pressurized thick-walled cylinder with symmetric radial cracks. Recently, Hantschel, Busch,Kuna and Maschke[150] modelled the plastic crack tip fields by the use of special singular elements and the introduction of the HRR fields. Leitao, Aliabadi, Rooke and Cook[151], [152] have used the BEM to simulate crack growth in presence of residual stress fields introduced by a cold expansion technique. Leitao and Aliabadi[153] demonstrated the efficiency of the BEM for evaluating several different nonlinear J-integrals.

Aliabadi and Cartwright[154] developed a BEM technique for the evaluation of a plastic zone size around a crack using the strip yield model. In their analysis a weight function formulation was used.

Only a few publications deal with mixed-mode elatoplastic problems. Rußwurm[155] addressed the mixed-mode problem using the crack Green's function approach, while Leitao, Aliabadi and Rooke[156] used an elastoplastic dual boundary element formulation. The problem of crack contact and elastoplastic behaviour was investigated by Leitao, Aliabadi and Rooke[157].

9 Non-metallic Materials

The boundary element method has been popular for analysis of cracks in geomechanical problems. In particular the indirect BEM formulation of displacement discontinuity method has been applied to many problems including movement of joints, fracture rocks and cracking due to earthquake response[158],[159]. One of the early application of BEM to simulate hydraulic fracturing is due to Clifton and Abou-sayed[160]. Later Sousa et. al[161] used DDM to study the coalescence and orientation of multiple fractures propagating from a wellbore which is not aligned with line of the principal stress directions. Hardy and Asgian[162] utilized the DDM to study the transient fracture fluid pressure and fracture width during representation of hydraulic fracture. A detailed review of formulations used to simulate hydraulic fracturing is presented by Asgian[163]. Hashida, et. al.[164] used the tension-softening model for the analysis of fracture processes of rock with particular reference to the effect of confining pressure on the fracture extension.

The application of BEM to the analysis of cracking in concrete is relatively new and there appears to be only a few publications on the subject. The use of DDM together with the fictitious crack model (FCM) was published by Harder[165], but no results were reported. Liang and Li[166] presented BEM analysis to simulate the nonlinear fracture zone in cementatious materials, using FCM. Later, Cen and Maier[167] used the multi-domain formulation with FCM to simulate crack growth in concrete. Saleh and Aliabadi[168],[169] used the dual boundary element method together with FCM for the analysis of both plain and reinforced concrete. Horii and Ichinomiya [170] used the Dugdale-Barenblatt model to analyse the fracture processes zone in concrete. They compared their results with the measurements of crack length and crack opening displacements obtained from a laser speckle technique, applied to a mortar and concrete specimen. Recently Alessandri and Dielo[171] presented a 2D BEM model for mode II failure of a plated concrete specimen. Pekau and Batta[172] developed a time domain BEM to study crack propagation in concrete structures subjected to seismic loadings.

Other important publications in non-metallic materials can be found for example in: Salvadurai and Au[173], Beer[174],Hashida[175] and Salvadurai[176]

10 Crack Identification

The boundary element method is an ideal technique for identification of internal cavities and cracks. Sakagami et. al[177] initially used a sophisticated trial and error procedure for the identification of internal straight line cracks in two-dimensional analysis, using electric potential measurements, i.e. satifying Laplace's equation. This multi-region scheme was subsequently modified with the inclusion of an optimization procedure for the identification of edge and corner cracks in 3D. Tanaka et.al.[178] also used a multi-region formulation with an elastodynamic BEM and an optimization procedure to identify two-dimensional internal cracks. Nishimura and Kobayashi[179] presented a crack identification method using a regularised form of the crack opening displacement method. This method has been implemented for potential and elastodynamic systems and used in both two- and three-dimensional problems. Mellings and Aliabadi[180],[181],[182] presented DBEM formulations for 2D problems in potential and elasticity and 3D potential.

11 Crack Growth Analysis

The first attempt to automatically model crack growth in mixed-mode conditions was by Ingraffea, Blandford and Ligget[183] for two-dimensional problems. They used the multi-region method together with maximum circumferential stress criterion to calculate the direction of crack growth. The extension of this multi-region method to 3D problems was presented by Grestle[184]. Crack growth processes in orthotropic materials was presented by Doblare et. al.[78]. They used the multi-region method together with quarter-point elements to simulate crack growth. The application of the multi-region method to dynamic crack growth was presented by Gallego and Dominguez [185]. In this paper the time-domain formulation was utilized together with quarter-point elements. Cen and Maier[167] also used the multi-region method to simulate crack growth in concrete structures. In their formulation the cohesive crack model was used to simulate the fracture process zone in concrete. The difficulty with the multi-region method is that the introduction of artificial boundaries to divide the regions is not unique, and thus cannot be easily implemented in an automatic procedure. In an incremental crack extension analysis, these artificial boundaries must be repeatedly introduced for each increment of crack extension.

Portela, Aliabadi and Rooke[186] and Mi and Aliabadi[187],[188] presented an application of the dual boundary element method (DBEM) to the analysis of mixed-mode crack growth in 2D and 3D linear elastic fracture mechanics. Crack growth processes were simulated with an incremental crack extension analysis based on the maximum principal stress criterion for 2D and minimum strain energy density criterion for 3D. In [186], for each increment of the crack extension, the DBEM was applied to perform a single region stress analysis and the J-integral technique used to compute the stress intensity factors. When the crack extension is modelled with new discontinuous elements, remeshing of the existing boundaries is not required because of the single-region analysis, an intrinsic feature of the DBEM. For surface breaking cracks in 3D a certain amount of remeshing is however required. An automatic procedure for this process has been developed by Aliabadi and Mi[189]. Salgado and Aliabadi[190] presented the application of DBEM to stiffened structures. They simulated crack growth in aircraft panels reinforced by stiffeners. The extension of DBEM to elastoplastic fatigue crack growth analysis has been presented by Leitao, Aliabadi and Rooke[191]. The extension of DBEM to static and transient thermoelastic crack growth is presented by Prasad, Aliabadi and Rooke[192],[193]. In these papers, the effect of thermal loading was studied on the crack growth and direction, both in phase and out-phase thermo-mechanical loads were considered. Fedelinski, Aliabadi and Rooke[194] have also extended the DBEM formulation in a time domain to the analyses of mixed-mode fast growing cracks. Sollero and Aliabadi[195] used DBEM for anisotropic materials to study mixed-mode crack growth in composite laminates. Latif and Aliabadi[169] developed a nonlinear cohesive crack model with DBEM for simulating cracking in both plain and reinforced concrete structures.

12 Indirect Boundary Element Formulations

Indirect boundary integral formulations for crack problems have been around for many years and appear in many branches of fracture mechanics. Hence, it will be extremely difficult to present a comprehensive review of them all. Most of the formulations are however, limited to isolated cracks in infinite domains. Nevertheless these solutions provide a valuable insight into the behaviour of cracks. Early applications of integral equation formulations to crack problems can be found in classical works of Sneddon[196] and Bibly and Eshelby[197]. Other important contributions can be found in the work of for example Erdogan and Gupta[198]. More recently, the concept of element discretization has been introduced to the method, which allows the solution of problems in finite domains (see Mira-Mohamad-Sadeg and Altiero[199], Le Van and Royer[200], Fares and Li[201]. The application of the method to kinked cracks and crack contact problems has been presented by Zang and Gudmundson[202].

12.1 Body Force Method

The body force method is one of the most prominent indirect BEM formulations. The method was originally proposed by Nisitani[203] for the solution of two-dimensional stress problems, and was later extended to the solution of notch[204] and crack problems[205]. As in BEM, the body force method uses the stress fields due to a point force in an infinite domain as fundamental solutions. The prescribed boundary conditions are satisfied by applying the body force along the imaginary boundaries in an infinite sheet and adjusting the force density so as to satisfy the boundary conditions. The boundaries of the problem are divided into a finite number of elements with unknowns defined at the mid-points of the elements. The application of the body force method to three-dimensional crack problems can be found in papers by Nisitani and Murakami[206] and Murakami and Nemat-Nasser[207]. For more recent advances in the body force method, readers should consult Nisitani[208].

12.2 Displacement Discontinuity Method

As discussed earlier, the standard application of the boundary integral formulation for crack problems has an inherent mathematical degeneracy due to the co-planar crack surfaces. To overcome this problem Crouch[209] proposed an indirect integral equation in which the unknowns are the displacement differences between the upper and lower crack surfaces. Furthermore the fundamental solutions are due to displacements discontinuities. The application of the displacement discontinuity method (DDM) to 3D problems have been reported by Weaver[210]. The extension of the method to dynamic crack problems has been presented by Das and Aki[211] using the time domain formulation in 2D and Das[212] for 3D. A comprehensive review of the recent developments in DDM with application to geomechanical problems is given by Mack[213].

The DDM is usually used to model the crack surfaces and not the non-cracked boundaries. For general crack problems DDM is complemented with another indirect integral equation known as the Fictitious Stress Method (FSM) to model the non-cracked boundaries. In the fictitious stress method, the real problem is transformed to an indirect problem in an infinite body and the outer boundary conditions including tractions and displacements are modelled by assumed distribution of loads (fictitious). Crouch

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and Starfield[214] give a detailed description of both formulations. Recently Wen, Aliabadi and Rooke[65],[67] developed a DDM and FSM in the Laplace transform space for both 2D and 3D problems. Other contributions can be found for example in Chan and Einstein[215]. Similar formulations to DDM have also been proposed by Cruse[216], Guidera and Lardner[217], Bui[218], Balas and Sladek[219] and Takaud, Koizumi and Shibuya[220].

13 Concluding Remarks

In this paper a review of boundary element formulations for fracture mechanics problems was presented. The review, although not exhaustive covers the important contributions and in particular those that have contributed to the method being used in practice. Much has been achieved during the last 25 years resulting in ever increasing research being carried out on this topic and greater achievements are anticipated in the future.

It is estimated that over one thousand papers have been written on the application of the BEM to fracture mechanics. For a complete list of BEM related references, readers should consult [221].

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