Evaluation of approximate methods for elastic-plastic analysis of notched bodies

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ABSTRACT

Predictions based on three different methods for estimating stress and strain at the root of a notch in an elastic-plastic body are compared with the results of finite element analysis for a uniaxially loaded double-edge notched plate under plane strain conditions. A generalisation of Neuber's rule based on equivalent stress and strain is shown to provide accurate results at nominal conditions ranging from linearly elastic to fully plastic. Estimates based on invariance of total strain energy density and strain energy density provide upper and lower bounds, respectively, to the energy density at the notch root.

INTRODUCTION

Through the development of advanced computational techniques and fast computers, accurate simulation of the mechanical behaviour of real components has become theoretically feasible. However, the actual accuracy of such simulations is highly dependent on an accurate description of material properties and loading conditions. Therefore, approximate methods based on the invariance of local energy density measures may provide attractive alternative routes to the calculation of stress and strain concentration in notched bodies. In particular, such simplified approximate methods enable the designer to carry out extensive parameter studies, thus obtaining a better understanding of the mechanical response of the component being investigated. The
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The purpose of the present paper is to determine the accuracy of three different approximate methods by comparing predictions with the results from finite element analysis.

APPROXIMATE METHODS

The basic concept of the approximate methods discussed here may be stated as follows: The ratio of a given strain energy measure at the root of a notch to the corresponding nominal measure is assumed to remain constant, independent of load level and material model. This constant ratio may be calculated for a linear elastic material. After determination of a consistent nominal state, it is possible to find the local state at the notch root for any non-linear material. In the current investigation, both nominal \((\sigma_n, \varepsilon_n)\) and local stress and strain \((\sigma, \varepsilon)\) are determined by FE analysis.

In general, it is possible to differentiate between the approximate methods by considering the definition of the invariant measure characterising the state at the notch root. The current presentation will concentrate on the following three approaches to the notch root estimation problem:

- Neuber's rule based on equivalent stress and strain.
- Total strain energy density invariance.
- Strain energy density invariance.

Neuber's rule [1] was originally developed for a notched prismatic body loaded in pure shear and characterised by an arbitrary non-linear constitutive model. The geometric mean value of the shear stress concentration factor, \(K_\tau = \tau/\tau_n\), and the shear strain concentration factor, \(K_\gamma = \gamma/\gamma_n\), is constant at any load state, and equal to the elastic stress concentration factor, \(K_t\):

\[
\sqrt{K_\tau \cdot K_\gamma} = K_t
\]  

For the case of multiaxial stress at the notch root, a generalisation of equation (1), already indicated in Neuber's original work [1] and later also proposed by Hoffman & Seeger [2], is the application of equivalent stress, \(\sigma_e\), and equivalent strain, \(\varepsilon_e\), given by:
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\[ \sigma_e = \sqrt{\frac{3}{2}} \sigma'_{ij} \sigma'_{ij}, \quad \varepsilon_e = \sqrt{\frac{2}{3}} \varepsilon'_{ij} \varepsilon'_{ij} \]

\( \sigma'_{ij} \) and \( \varepsilon'_{ij} \) denote the deviatoric stress and strain respectively. Neuber's rule may then be written as:

\[ \sqrt{\frac{\sigma'_e}{\sigma'_{en}}} \cdot \frac{\varepsilon'_e}{\varepsilon'_{en}} = \sqrt{K_{\alpha e} \cdot K_{\varepsilon e}} = K_{\varepsilon e} \quad (2) \]

Thus, in determining the equivalent stress and strain at the root of a notch, only two equations have to be considered: (i) the Neuber equation given above and (ii) the actual uniaxial stress-strain curve. The elastic stress concentration factor and the nominal load then yield an approximate solution to the notch root stress and strain.

Moftakhar and Glinka [3] propose a generalisation of Neuber's rule by assuming the total strain energy density, \( \sigma : \varepsilon \), to be invariant:

\[ \sqrt{\frac{\sigma : \varepsilon}{\sigma' : \varepsilon'}} = C_{UB} \quad (3) \]

According to [3], equation (3) yields an upper bound to notch root total strain energy density, whereas equation (4) below yields a lower bound to notch root strain energy density, \( \int \sigma : d\varepsilon \):

\[ \sqrt{\frac{\int \sigma : d\varepsilon}{\int \sigma' : d\varepsilon'}} = C_{LB} \quad (4) \]

The complete formulation of Moftakhar and Glinka provides estimates to all stress and strain components at the notch root. In the current investigation these components will not be considered explicitly.

Large scale yielding

The methods for estimation of notch root plastic response are usually limited to cases of localised plasticity. They can be extended to comprise gross plasticity by applying the actual constitutive equation when calculating the nominal strain, \( \varepsilon_n = f(\sigma_n) \), rather than the linear elastic approximation.
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NUMERICAL ANALYSIS

The accuracy of the approximate methods presented is investigated by carrying out a complete FE analysis of a notched configuration with nominal conditions ranging from linearly elastic to fully plastic.

Geometry and loading
A uniaxially loaded, finite-width plate under plane strain conditions with a semi-circular double-edge notch is considered.

A dense mesh is used for the FE calculations with 18 (second-order) elements along the quarter-circle contour and 24 elements along the net-section.

Material
A power-law hardening material according to Ramberg-Osgood has been assumed:

$$\varepsilon = \frac{\sigma}{E} + B\sigma^n$$

Without losing any generality, equation (5) may be rewritten by introducing a characteristic strain, $\varepsilon_0$, and a corresponding characteristic stress, $\sigma_0 = E\cdot\varepsilon_0$. Making a suitable choice of $\varepsilon_0$ yields:
The calculation results obtained using this normalised constitutive equation may be scaled to represent any choice of $E$, $\varepsilon_0$ and $B$. Therefore, only the dimensionless material parameters $n$ and $\nu$ (Poisson's ratio) are significant in determining normalised stresses and strains. To reflect typical hardening exponents, $n$ is chosen to be 5, 10 and 20 in the current analysis. For Poisson's ratio, only the case of $\nu=0.3$ has been considered.

Validity of results
The practical applicability of the current results to cases of gross plasticity is limited since small displacements have been assumed. The results will however be significant to cases of steady state creep due to the analogy between plastic strain and visco-plastic strain rate.

RESULTS AND DISCUSSION

FE results are presented in Figure 2 for a plane strain finite width plate configuration with a semi-circular double-edge notch (Figure 1).

Equivalent stress and strain concentration factors based on net section equivalent stress and strain, are calculated from the FE results. They are compared with the stress and strain concentration factors obtained by using Neuber's rule as given in equation (2).

For the configuration presented, it is clearly seen that Neuber's rule gives close estimates to stress and strain concentration factors through the whole range of loading. Stress concentration factors are predicted without noticeable inaccuracy. Equivalent strain concentration factors are generally predicted on the safe side with a maximum deviation from FE results of 12.5% ($n=20$, $\sigma_{en}/\sigma_0 = 0.7$). At general yielding, the deviation between FE results and estimated values of $K_{\varepsilon_0}$ is less than 1.5%.

\[
\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \left( \frac{\sigma}{\sigma_0} \right)^n
\]
Figure 2: Equivalent stress and strain concentration factors at the notch root for a uniaxially loaded semi-circular double-edge notched plate under plane strain. Comparison between results from FE analysis and predictions by Neuber's rule.

To make an assessment of the accuracy of the different approximate models, the left hand sides of equations (2) through (4) have been computed:

\[ F_N = \frac{\sigma_e \cdot \varepsilon_e}{\sigma_{en} \cdot \varepsilon_{en}} \]  
\[ F_{UB} = \sqrt{\frac{\sigma \cdot \varepsilon}{\sigma_n \cdot \varepsilon_n}} \]  
\[ F_{LB} = \sqrt{\frac{\int \sigma : d\varepsilon}{\int \sigma_n : d\varepsilon_n}} \]
The ratios of the elastically calculated estimates $K_{te}$, $C_{UB}$ and $C_{LB}$ to the actually computed values $F_N$, $F_{UB}$ and $F_{LB}$ have been plotted as functions of $\sigma_{en}/\sigma_0$ in Figure 3.

![Figure 3: Deviations between predictions of approximate methods and FE results for a uniaxially loaded semi-circular double-edge notched plate under plane strain.](image)

Upper and lower bound ratios are bounding the ratio associated with Neuber's rule. Through the whole loading range, equation (4) gives a strain energy density at the notch root, which is less than the one obtained in the FE analysis, whereas equation (3) predicts a higher total strain energy density than the FE analysis.

The generalisation of Neuber's rule based on equivalent stress and strain yields estimates of the Neuber product, $K_\sigma K_e$, which vary from being larger than to being slightly smaller than what is found in the FE analysis.

**CONCLUSION**

Neuber's rule provides a close estimate to notch root equivalent stress and strain at any load level. The application of total strain energy density and strain energy density criteria to the notch root response produce results, which bound the solutions obtained with the current application of Neuber's rule.
The presented results demonstrate how properly applied approximate methods can provide accurate estimates to notch root stress and strain.

REFERENCES

