Strain energy release rate calculation from 3-D singularity finite elements

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ABSTRACT

Two expressions for the strain energy release rate for three-dimensional singularity finite elements are derived based on the Irwin’s virtual crack closure method. The strain energy release rate for the three modes of fracture mechanics can be separately calculated from the nodal forces ahead of the crack front and the opening displacements behind it. The material is assumed to be isotropic. The stress distribution is written in terms of the eight nodal forces of the crack front element, and the crack opening displacement function is expressed in terms of the nodal opening displacement behind the crack front. The derived formula is a bit complicated because it involves eight nodal forces and five opening displacements. Therefore, a modification is made to get a simpler formula by eliminating some of the nodal forces from the original one. Comparisons with some problems from the literature indicate that the derived formulæ are accurate for the computation of the strain energy release rate from 3-D singularity finite elements.

INTRODUCTION

The strain energy release rate is one of the most important fracture mechanics parameters for cracks in isotropic and orthotropic materials. Moreover, in the case of an orthotropic material, it was found that the energy release rate is more convenient than the stress intensity factor [1]. For example, in composite laminates, the delamination onset loads are affected by the ply thickness. This can be analyzed by using the strain energy release rate as a fracture mechanics criterion.

In some circumstances, a three-dimensional finite element analysis should be performed when neither plane stain nor plane stress conditions prevail. One has also to realize that in a composite material, an out of plane deformation can take place due to in plane loading, so eventually mode III effects have to be considered [2].

The crack closure integral method was first proposed by Irwin [3]. Many authors used this method in their finite element analysis by using the stresses ahead of the crack tip and the opening displacements behind it. Rybicki and Kanninen [4] developed a formula for the strain energy release rate for a 2-dimensional 4-node (linear) element based on the crack closure integral method. Raju [5] extended the method for the 2-dimensional case to include 8-node (parabolic), 12-node (cubic) non-singular elements, and quarter point and cubic singularity
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Sethuraman and Maiti [6] used the method by assuming that the variation of the stresses is one order less than that of the displacements. They got simple expressions for the 8-node singular (quarter point singularity) element, 4-node and 8-node quadrilateral non-singular elements. For cracks along the interface in a bi-material problem, although the strain energy release rate formulae based on the crack closure integral method neglect the oscillation of the stresses near the crack tip, acceptable results are obtained.

Sun and Jih [7] used the formula developed by Rybicki and Kanninen [4] in their finite element analysis to predict the strain energy release rates for mode I and mode II in the case of a crack along a bi-material interface. They got good agreement between the crack-closure integral method and the analytical solution which includes the oscillation terms for different crack tip element sizes.

The advantages of using the crack closure integral method are the need to solve the finite element problem only once, and the separation of the energy release rate for different modes in a mixed mode problem.

The derivation presented here is close to that done by Raju [5] for the two-dimensional finite element idealization. The procedure is applied to the 20-node singularity elements. A modification is done to the derived formula to get a simpler expression. Then, two problems are analyzed in order to evaluate the derived formulae:

MATHEMATICAL DERIVATION

3-D 20-node singular element

Figure (1) shows 3-D 20-node brick elements (1a) and 3-D 15-node prism-shaped elements (1b) around a crack front. The desired singularity at the crack front is achieved by moving the mid-side node to the quarter position [8]. The finite elements are assumed to be symmetric about the crack front (x, y and z). The normal stress distribution \( \sigma_y \) ahead of the crack front in the plane (x,y) is assumed to be as follows:

\[
\sigma_y = \frac{A_1}{\sqrt{x}} + A_2 x + A_3 y + A_4 \sqrt{x} + A_5 y + A_6 \sqrt{x} + A_7 y^2 + A_8 \sqrt{x}
\]

The equation involves 8 constants \( A_1 \) to \( A_8 \) which could be written in terms of the 8 nodal forces of the crack tip element ahead of the crack, i.e. nodes i,j,k,l,p,m,n,o. To get this relation, the work done by the stress distribution \( \sigma_y \) is put equal to that done by the 8 nodal forces in the z-direction, i.e.;

\[
\frac{1}{2} \int_{0}^{\Delta} \sigma_y(x,y) w(x,y) \, dx\,dy = \frac{1}{2} (F_x w_i + F_y w_j + F_z w_k + F_x w_l + F_y w_p + F_z w_m + F_x w_n + F_y w_o)
\]
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where t and Δ are the thickness and the length of the crack tip element, respectively. w(x,y) is the interpolated displacement function in the z-direction:

\[ w(x,y) = N_i w_i + N_j w_j + N_k w_k + N_m w_m + N_n w_n + N_o w_o \]  

(3)

where \( N_i, N_j, N_k, N_m, N_n, N_o \) are the shape functions of the element in the (x,y) plane, and they are written in terms of the natural coordinates \((ξ, η)\).

For a singular element, where nodes n and p are at the quarter position, and node m and o are at the mid-side position, the relations between \((ξ, η)\) and \((x,y)\) are in the form:

\[ ξ = 2 \left( \frac{x}{Δ} - 1 \right) \quad \eta = 2 \frac{y}{l} - 1 \]  

(4)

From equation (2), the force at node i can be calculated by the following integration:

\[ F_i = \int \int q_i N_i \, dx \, dy \]  

(5)

Similar expressions can be written for the other nodal forces. Substituting equation (1), (3) and (4) into equation (5) (for all the nodal forces) and performing the integration over (x,y), yields:

\[ [F] = [Q][A'] \]

where

\[ [F] = [F_x F_y F_z F_{x'} F_{y'} F_{z'}] \]

\[ [A'] = [A_{11} A_{12} A_{13} A_{21} A_{22} A_{23} A_{31} A_{32} A_{33}] \]

\[ [Q] = \frac{1}{180} \begin{bmatrix} -30 & -20 & -10 & -13 & -10 & -6 & -5 & -3 \\ -30 & -10 & 0 & -3 & -10 & 3 & 3 & -3 \\ -30 & -10 & -10 & -3 & -20 & -7 & -6 & -13 \\ -30 & -20 & -10 & -13 & -20 & -6 & -6 & -13 \\ 120 & 60 & 20 & 36 & 40 & 10 & 12 & 20 \\ 120 & 40 & 20 & 20 & 60 & 12 & 10 & 36 \\ 120 & 60 & 40 & 36 & 80 & 30 & 24 & 60 \\ 120 & 80 & 40 & 60 & 60 & 24 & 30 & 36 \end{bmatrix} \]  

(6)

To express the constants \([A']\) in terms of the nodal forces, the
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The strain energy release rate derived from the Irwin’s crack closure integral method [3] assumed that for an infinitesimal crack extension $\Delta$, the opening displacements behind the new crack tip are approximately the same as those behind the original one. Thus, the work necessary to extend the crack by $\Delta$ is equal to that necessary to close it by $\Delta$. Based on this assumption the energy release rate is given by:

$$G_i = \lim_{\Delta \to 0} \frac{1}{2\Delta} \iint_0^\Delta w_r(r,y) \sigma(\Delta-r,y) \, dr \, dy$$

(8)

where $w_r(r,y)$ is the crack opening displacement at the position $(r,y)$ behind the crack front. As the crack tip element becomes smaller, the limit can be omitted, and $G$ approaches the exact value. The opening displacement function $w_r(r,y)$ can be determined at any position $(r,y)$ from the nodal opening displacements at nodes $(q,s,w,u,v)$ and their shape functions as:

$$w_r(r,y) = N_q w_q' + N_s w_s' + N_w w_w' + N_u w_u' + N_v w_v'$$

(9)

Where $w'$ is the nodal opening displacement in the $z$-direction. Note that the displacements at nodes $(i,l,m)$ are omitted from equation (9) because their opening displacements are zero and they will not contribute in the $G$ formula.

Substituting equations (1) and (9) into equation (8), then using relation (7), and solving for $G_i$, leads to the following expression:

$$G_i = \frac{1}{2\Delta} \left\{ F(x(C_1 w_1' + C_2 w_2' + C_3 w_3' - C_4 w_4')) + F(x(C_4 w_4' + C_5 w_5' + C_6 w_6' - C_7 w_7')) + F(x(C_5 w_5' + 2C_2 w_2' + C_3 w_3' + 2C_7 w_7') + F(x(C_6 w_6' + 3C_6 w_6' + C_7 w_7' + C_1 w_1' + 2C_4 w_4') + F(x(C_7 w_7' - 3C_2 w_2' - C_3 w_3' - C_1 w_1' + 2C_6 w_6') + F(x(C_1 w_1' - C_2 w_2' - C_3 w_3' - C_4 w_4' + C_5 w_5' + C_6 w_6' + C_7 w_7' + C_1 w_1')) + F(x(C_4 w_4' + C_5 w_5' + C_6 w_6' + C_7 w_7' + C_1 w_1' + C_2 w_2' + C_3 w_3' + C_4 w_4') + F(x(C_5 w_5' + C_6 w_6' + C_7 w_7' + C_1 w_1' + C_2 w_2' + C_3 w_3') + F(x(C_6 w_6' + C_7 w_7' + C_1 w_1' + C_2 w_2' + C_3 w_3' + C_4 w_4')) + F(x(C_7 w_7' + C_1 w_1' + C_2 w_2' + C_3 w_3' + C_4 w_4' + C_5 w_5' + C_6 w_6' + C_7 w_7')) \right\}$$

(10a)
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where

\[
C_1 = \frac{33\pi}{4} - 26 \\
C_2 = \frac{25\pi}{24} - \frac{10}{3} \\
C_3 = \frac{31\pi}{12} + \frac{26}{3} \\
C_4 = \frac{9\pi}{4} - 6 \\
C_5 = -\frac{13\pi}{8} + \frac{8}{3} \\
C_6 = -\frac{29\pi}{8} + 12 \\
C_7 = -\frac{67\pi}{16} + 13 \\
C_8 = \frac{21\pi}{4} - 16 \\
C_9 = -\frac{91\pi}{12} + 19 \\
C_{10} = -\frac{67\pi}{32} - 6 \\
C_{11} = -\frac{21\pi}{4} + 17 \\
C_{12} = \frac{37\pi}{12} - \frac{26}{3} \\
C_{13} = -\frac{173\pi}{48} + \frac{34}{3} \\
C_{14} = -\frac{9\pi}{8} + 4 \\
C_{15} = \frac{39\pi}{72} - \frac{5}{6} \\
C_{16} = \frac{19\pi}{12} - \frac{14}{3} \\
C_{17} = -\frac{303\pi}{144} - \frac{19}{3}
\]

Similar expressions can be written for mode II and mode III by replacing \( F_\gamma \) by \( F_\delta \) or \( F_\zeta \) and \( w \) by \( u \) or \( v \), respectively.

Note that, for the prism element, the nodal forces in equation (12) are calculated from the elements \((I, J, K, L)\) around the crack front.

Modification of the G formula

The G formula (equation (10)) is a bit complicated because it involves 17 constants due to the presence of all nodal forces of the crack front element ahead of the crack. Besides, for mixed mode condition, the G formula should be modified to take into account that the nodal forces \((F_\gamma, F_\delta, F_\zeta)\) calculated from the upper and lower crack faces are not equal in magnitude and opposite in sign. This is due to the fact that for mixed mode conditions, the deformations of the crack faces are neither symmetric nor antisymmetric about the crack front plane. Raju [9] has shown that this modification can be done by separating the amount of work required to close each crack surface. Thus, the terms in the G formula (equation (10)) concerning the nodal forces \((F_\gamma, F_\delta, F_\zeta)\) can be expressed in terms of the forces calculated at the upper and lower crack surfaces and the displacement of the nodes behind the crack front relative to the nodes at the crack front. Another way to capture this problem, is trying to eliminate the three nodal forces \((F_\gamma, F_\delta, F_\zeta)\) from equation (10). This could be done, if those nodal forces are written in terms of the remaining nodal forces. By putting \( A_\eta, A_\zeta, A_\xi \) equal to zero in equation (7), the requested relations can be found as:

\[
F_\gamma = F_\xi + \frac{1}{2}F_\zeta - \frac{1}{2}F_\eta \\
F_\delta = F_\xi + \frac{1}{2}F_\zeta - \frac{1}{2}F_\eta \\
F_\zeta = -2F_\eta - 2F_\xi
\]

Substituting those relations into equation (10), the following modified G formula is obtained:
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\[
G_t = \frac{1}{2\pi a} \{ F_z (C_1 w'_z - C_2 w'_t - \frac{C_3}{2} w'_t) + F_y (C_2 w'_y - 2C_4 w'_t) + F_z (C_1 w'_z + C_2 w'_t + \frac{C_3}{2} w'_t) \}
\]

where

\[
C_1 = 6\pi - 20 \quad C_2 = \pi - 4 \\
C_3 = \pi - 2 \quad C_4 = \frac{5\pi}{4} + 4
\]

This equation is simpler than equation (10). It involves only 4 constants, and it is valid for mixed mode conditions. Similar expressions can be obtained for mode II and III by replacing \( F_z \) by \( F_x \) or \( F_y \) and \( w \) by \( u \) or \( v \), respectively.

INVESTIGATION OF ACCURACY

Two problems are analyzed to illustrate the accuracy of the derived formulae. The first problem is taken from reference [10], where a centre-cracked tension specimen problem is solved analytically for different crack length to width ratio (\( a/b \)). The specimen is shown in figure (2). In the current 3-D analysis, the nodal displacements in the \( y \)-direction at the top and the bottom surfaces of the plate are set to zero to represent a 2-D plane strain situation. The specimen is subjected to remote load \( S \). Only one-quarter of the specimen has to be analyzed due to symmetry. The finite element mesh is shown in figure (3). The crack tip element size is \( 0.0625 \) times \( a \), where \( a \) is the crack length. The comparison between the results of reference [10] and the current analysis is presented in table (1). The \( \beta \) value which is used in the comparison, is given by:

\[
\beta = \frac{EG_t}{S^2 a (1-\nu)}
\]

Two different \( G \) values for the 3-D analysis are calculated using equation (10) and (12).

<table>
<thead>
<tr>
<th>( a/b )</th>
<th>Ref. [10]</th>
<th>Eq. (10)</th>
<th>Eq. (12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>1.1223</td>
<td>1.1016</td>
<td>1.11382</td>
</tr>
<tr>
<td>0.4</td>
<td>1.236</td>
<td>1.20318</td>
<td>1.21616</td>
</tr>
<tr>
<td>0.5</td>
<td>1.4139</td>
<td>1.37737</td>
<td>1.39216</td>
</tr>
<tr>
<td>0.6</td>
<td>1.70119</td>
<td>1.6649</td>
<td>1.68296</td>
</tr>
<tr>
<td>0.7</td>
<td>2.20284</td>
<td>2.1728</td>
<td>2.19644</td>
</tr>
</tbody>
</table>
The comparison between those two values indicates that the modified formula for singular element is more accurate than its consistent counterpart. In general, the results show good agreement between the 3-D analysis and the reference values. The second problem is a single edge notched specimen, taken also from reference [10]. The problem configuration is shown in figure (4). Due to symmetry, only half of the specimen is modeled with the appropriate symmetric boundary conditions. The finite element mesh is the same as in the first problem (figure (3)). The comparison with the accurate results from reference [10] is presented in table (2). The same remarks as in the first problem can be observed.

Table (2): Single edge notched specimen

<table>
<thead>
<tr>
<th>a/b</th>
<th>Ref. [10]</th>
<th>Eq. (10)</th>
<th>Eq. (12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>2.7652</td>
<td>2.77622</td>
<td>2.8076</td>
</tr>
<tr>
<td>0.4</td>
<td>4.4377</td>
<td>4.4066</td>
<td>4.4558</td>
</tr>
<tr>
<td>0.5</td>
<td>8.007</td>
<td>7.8232</td>
<td>7.9135</td>
</tr>
<tr>
<td>0.6</td>
<td>16.2408</td>
<td>15.8794</td>
<td>16.0632</td>
</tr>
<tr>
<td>0.7</td>
<td>40.4623</td>
<td>39.0537</td>
<td>39.5376</td>
</tr>
<tr>
<td>0.8</td>
<td>143.736</td>
<td>134.747</td>
<td>136.76</td>
</tr>
</tbody>
</table>

CONCLUSION

Strain energy release rate formulae for the 3-D, 20-node singular finite element were derived based on the Irwin’s crack closure integral method. A modified formula which is simpler than its counterpart was suggested. Two problems were analyzed to evaluate the derived formulae. The results were compared to those from the literature. These comparisons indicate that the derived G formulae are accurate for the computation of the strain energy release rate from 3-D singularity finite elements.

REFERENCES


9. Raju, I.S. 'Simple formulae for strain energy release rates with higher order and singular finite elements', National Aeronautics and Space Administration, NASA CR 178186, 1986

Figure 1: a) 3-D 20-noded brick singular element, b) 3-D 15-noded prism-shaped singular element.
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Figure 2: Centre-cracked tension specimen.

Figure 3: Finite element mesh.

Figure 4: Single edge notched specimen.