Modelling the dynamic behaviour of a ball-screw system taking into account the changing position of the ball-screw nut

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Abstract

An estimation of the dynamic behaviour of ball-screw systems is particularly important in the industrial processes where significant cost savings can be achieved by optimising the drive operation. The dynamic behaviour of a ball-screw system is modelled using a finite element approach in this paper. The nut exerts a significant lateral restraint on transverse vibration of the screw in the case of a ball-screw. The effect of this vibration varies with time as the nut moves. This implies that the natural frequencies of a ball-screw system change with time. The method described models the dynamic behaviour of a ball-screw taking the moving nut into account. The analysis accuracy was improved by integrating the accumulated data (damping ratios, natural frequencies, stiffness) acquired from evaluation experiments of each element and experimental modal analysis. The simulated results have been proved satisfactory when compared with measured data taken from a CNC machine tool. The model, which can simulate a variety of non-linear mechanical behaviour in a ball-screw system, also predicts the amount of heat generated. A better understanding of ball-screw dynamics was gained through the modelling process and it is hoped to use the energy dissipation features in predicting thermal behaviour of ball-screws.

1 Introduction

A ball screw system is often used as a driving unit for machine tools because of its high rigidity and accuracy. The dynamic behaviour of a ball-screw, like that of many other mechanical systems can be modelled using a finite element
approach, a highly proven technical procedure. The available methods give the natural frequencies and mode shapes of a system [1-3] or the amplitude of "steady state" vibration for a particular form of excitation [4]. These methods do not describe adequately those aspects of the ball-screw dynamics which change with the position of the nut along the screw. These include:

- the nut produces a significant lateral restraining effect on transverse vibration of the screw;
- the "coupling point" between the torsional and axial movement of the screw and lateral and tilting movement of the part of the machine to which the nut is attached.

These effects vary with time as the nut moves so the natural frequencies of a ball-screw system change with time.

Other aspects considered in order to perform a reliable simulation of the ball-screw dynamics are the contact of each ball-screw element to the adjacent elements and boundary conditions. The modal parameters (damping ratios, natural frequencies) and non-linearities such as Coulomb friction have been determined from experimental results [5].

The proposed dynamic model of ball-screw is validated by comparing simulated Bode diagrams for white noise input with measured ones for the same stimuli conditions at the machine.

2. Dynamics of a ball-screw with a moving nut

At first the possibility of an analytical solution in terms of beam theory was considered [6]. However the natural tendency was to split the beam representing the screw into two at the nut. This generated a fundamental problem because at this point the boundary conditions and continuity are difficult to define since the split point is changing with time.

![Figure 1: Ball-screw model](image)

In order to overcome this disadvantage, the following approach was settled upon. Consider the ball-screw split into S equal portions, thus generating S+1 nodes. If the screw has length l, the first and the S+1th element have
length \( l/2S \) and all the intermediate elements length \( l/S \). The mass of the screw is considered to act at the nodes and the stiffness is considered to operate in the \( S \) elements. The bearings are modelled by restraints applied initially at the end of the screw (see Figure 1).

The nut is modelled as a single node at which the mass is considered to act, and the stiffness between the nut and the screw is considered as a single value \( k \). In principle, the nut is considered to be connected to all ball-screw nodes. A stiffness matrix can be set up for the ball-screw system as follows:

\[
K = \begin{bmatrix}
K + \xi_1 k & -K & 0 & \ldots & 0 & 0 & -\xi_1 k \\
-K & 2K + \xi_2 k & -K & \ldots & 0 & 0 & -\xi_2 k \\
0 & \ldots & -K & 2K + \xi_3 k & \ldots & 0 & 0 & -\xi_3 k \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 2K + \xi_5 k & -K & -\xi_5 k \\
0 & 0 & 0 & \ldots & \ldots & K + \xi_{S+i} k & -\xi_{S+i} k \\
-\xi_1 k & -\xi_2 k & -\xi_3 k & \ldots & -\xi_5 k & -\xi_{S+i} k & k
\end{bmatrix}
\]  

where \( K \) = the stiffness of a single element of the ball-screw

\( \xi_i \) = the factors whose value depends on the position of the nut.

In order to incorporate the nut stiffness only once, the values of the factors \( \xi_i \) must satisfy the following condition:

\[
\sum_{i=1}^{S+1} \xi_i = 1
\]  

(2)

In addition to the terms in Equation (1), the stiffnesses \( K_{B1} \) and \( K_{B2} \) must be added to the diagonal to represent the restraining effects of the bearings.

If the ball-screw is at a position \( x_n \) between nodes \( j-1 \) and \( j \), the values of \( \xi_{j-1} \) and \( \xi_j \) will be given by the following relation:

\[
\xi_{j-1} = \frac{x_j - x_n}{l/S} \\
\xi_j = \frac{x_n - x_{j-1}}{l/S}
\]  

(3a)

For \( i \neq j-1 \) and \( i \neq j \),

\[
\xi_i = 0
\]  

(3b)

In a similar way, a matrix \( C \) of damping coefficients can be built up with terms of \( C_{B1} \) and \( C_{B2} \) being added to the diagonal to represent the viscous drag of the bearings. The solution of the moving ball-screw problem therefore reduces to the solution of the following equation:

\[
M\ddot{x} + C\dot{x} + Kx = f(t)
\]  

(4)

where \( M \) = the mass matrix, \( x \) = the displacement array and \( f \) = the force array. The matrices \( C \) and \( K \) can change with time.
3. Solution method

The method being used is a power series solution of the form:

$$ x = \sum_{i=0}^{\infty} a_i t^i $$

(5)

Differentiating with respect to time gives the velocity:

$$ \dot{x} = \sum_{i=0}^{\infty} (i+1) a_{i+1} t^i $$

(6)

A further differentiation gives the acceleration:

$$ \ddot{x} = \sum_{i=0}^{\infty} (i+1)(i+2) a_{i+2} t^i $$

(7)

Let us regard the force as a constant vector for the time interval over which we seek a solution, and let \( C \) and \( K \) also remain constant. This is reasonable provided that the time interval over which the solution is sought is short compared to the scale over which changes in \( f, C \) and \( K \) take place. In this case, substituting Equations (5)-(7) with constant terms into Equation (4) yields:

$$ M \times (0+1)(0+2) \times a_{0+2} + C \times (0+1) \times a_{0+1} + K \times a_0 = f $$

(8)

where

$$ a_2 = -\frac{M^{-1}C}{2} a_1 + \frac{M^{-1}K}{2} a_0 + \frac{M^{-1}f}{2} $$

(9)

Comparing coefficients for \( t^k \) gives

$$ M \times (k+1)(k+2) \times a_{k+2} + C \times (k+1) \times a_{k+1} + K \times a_k = 0 $$

where

$$ a_{k+2} = -\frac{M^{-1}C}{k+2} a_{k+1} - \frac{M^{-1}K}{(k+1)(k+2)} a_k $$

(10)

or

$$ a_k = -\frac{M^{-1}C}{k} a_{k-1} - \frac{M^{-1}K}{(k-1)k} a_{k-2} $$

(11)

Thus, all the other coefficients \( a_k \) are calculable when the initial displacement vector \( a_0 \) and the initial velocity vector \( a_1 \) are given. Furthermore, it can be observed that the higher order coefficients are derived from the lower order ones by dividing by larger and larger numbers. The expectation that the series will converge to a finite solution results from comparing the previous conclusion with the power series expansions for \( e^x \), \( \sin x \) and \( \cos x \).

Physical constraints on the system require that \( f(t) \) has to be finite with finite discontinuities (the worst case). This signifies that the "roughest ride" that the system is going to experience, is one with finite discontinuities in acceleration. Therefore the velocity will always be continuous and the displacement "smooth". Such mathematically "well behaved" functions are likely to cause relatively little trouble in computation.

A program implementing the above described algorithm and using typical values of a Beaver VC35 CNC machine tool was written in C language.
main conclusion generated by analysing the calculated results was that \( a_k \) diverged significantly before they converged. However, since the time interval considered was small \( T \) converged rapidly. To avoid problems with multiplying very large numbers by very small ones, the displacement \( d \) at the end of a time interval \( T \) can be obtained by using the following series:

\[
d = x = \sum_{i=0}^{\infty} b_i, \text{ where } b_i = a_i T^i
\]

(12)

Considering \( d_0 \) to be the displacement and \( v_0 \) to be the velocity at the beginning of the time interval, \( b_0 \) and \( b_1 \) are calculable from the initial conditions as follows:

\[
b_0 = d_0 \text{ and } b_1 = v_0 T
\]

(13)

Multiplying Equation (9) by \( T^2 \) gives:

\[
b_2 = a_2 T^2 = \frac{M^{-1}C}{2} a_1 T^2 - \frac{M^{-1}K}{2} a_0 T^2 + \frac{M^{-1}f}{2} T^2
\]

(14)

\[
= -\frac{M^{-1}CT}{2} a_1 T - \frac{M^{-1}KT^2}{2} a_0 T^0 + \frac{M^{-1}f}{2} T^2
\]

and multiplying Equation (11) by \( T^2 \) yields:

\[
b_k = a_k T^k = -\frac{M^{-1}C}{k} a_{k-1} T^{k-1} - \frac{M^{-1}K}{(k-1)k} a_{k-2} T^{k-2}
\]

(15)

Making the substitutions \( \alpha = -M^{-1}K t \) and \( \beta = -M^{-1}C t \) into Equations (14) and (15) gives:

\[
b_2 = \frac{\beta}{2} b_1 + \frac{\alpha}{2} b_0 + \frac{M^{-1}f}{2} T^2
\]

(16)

\[
b_k = \frac{\beta}{k} b_{k-1} + \frac{\alpha}{(k-1)k} b_{k-2}
\]

(17)

where \( b_k \) = column vectors with \( N \) elements

\[\alpha, \beta = N \times N \text{ matrices}\]

\[N = \text{ the order of } M, C \text{ and } K.\]

Thus, \( b_2 \) can be derived from \( b_0 \) and \( b_1 \) (set by the initial conditions) and knowing the force, and \( b_k \) can be derived from \( b_{k-1} \) and \( b_{k-2} \) for \( k \geq 3 \).

The ball-screw velocity can be calculated starting from Equation (6):
Thus
\[ \mathbf{c}_i = (i + 1) \mathbf{a}_{i+1} T^i \]
\[ \mathbf{c}_0 = (0 + 1) \mathbf{a}_{0+1} T^0 = \mathbf{a}_1 = \mathbf{v}_0 \]
\[ \mathbf{c}_1 = (1 + 1) \mathbf{a}_{i+1} T^1 = \frac{2a_z T^2}{T} = \frac{2b_z}{T} \]

Substituting for \( a_{i+1} \) from Equation (10) into Equation (19) gives a recurrence relation for \( \mathbf{c}_k \):
\[
\mathbf{c}_k = (k + 1) \mathbf{a}_{k+1} T^k = \frac{(k + 1) \mathbf{M}^{-1} \mathbf{C}}{(k + 1)} \mathbf{a}_k T^k \quad \frac{(k + 1) \mathbf{M}^{-1} \mathbf{K}}{k(k + 1)} \mathbf{a}_{k-1} T^{k-1} = - \frac{\mathbf{M}^{-1} \mathbf{C} T}{k} \mathbf{k} \mathbf{a}_k T^{k-1} \quad \frac{\mathbf{M}^{-1} \mathbf{K} T^2}{(k - 1) k} \mathbf{a}_{k-1} T^{k-2} 
\]
\[
\frac{\beta}{k} \mathbf{k} \mathbf{a}_k T^{k-1} \quad \frac{\alpha}{(k - 1) k} \mathbf{a}_{k-1} T^{k-2} \]
\[
\mathbf{c}_{k-1} = \frac{\beta}{k} \mathbf{c}_{k-1} + \frac{\alpha}{(k - 1) k} \mathbf{c}_{k-2}
\]
where \( \mathbf{c}_k \) = column vectors with \( N \) elements.

It is now possible to define the state \( \{d, v\} \) at the end of a time step of duration \( T \) in terms of its state \( \{d_0, v_0\} \) at the beginning.

4. Development of computer programs to solve the 6 degree of freedom (axial, transverse, torsional and tilt) case

The dynamic behaviour of a ball-screw system with a moving nut is calculated by starting at some initial state \( \{d_0, v_0\} \) and solving Equation (4) for a time step of duration \( T \). The final state given by Equations (12) and (18) is used as the initial state of the next time step and so on.

Progress of the nut is accounted for by redefining \( \mathbf{K} \) and \( \mathbf{C} \) at each step, which means that \( \alpha \) and \( \beta \) in Equation (16) and recurrence relations (17) and (22) are also redefined. This process is repeated for as long as necessary.

A computer program was written in C to generate the displacement and velocity time histories of all the nodes in the ball-screw system. To avoid writing graphic routines for C it was decided to write the results to file and use Matlab to plot them. In addition to the modelling the ball-screw as described in Section 2, terms were included to take account of:

- a controller, an electric motor and mechanical transmission (belt drive);
- Coulomb friction in the bearings;
- Coulomb friction in the slideways carrying the part to which the nut is attached;
- ball-screw backlash.
The transfer functions of the controller and experimental values for many of the physical variables were determined by Pislaru [5]. The K terms from Equation (1) are in fact sub-matrices based on beam theory:

\[
K = \begin{bmatrix}
\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI}{l^3} & 0 & 0 & 0 & -\frac{6EI}{l^2} \\
0 & 0 & \frac{12EI}{l^3} & 0 & \frac{6EI}{l^2} & 0 \\
0 & 0 & 0 & \frac{GK}{l} & 0 & 0 \\
0 & 0 & \frac{6EI}{l^2} & 0 & \frac{EI}{l} & 0 \\
0 & \frac{6EI}{l^2} & 0 & 0 & \frac{EI}{l} & 0
\end{bmatrix}
\]

(23)

where \( E \) = the elastic modulus, (N/m\(^2\))

\( A \) = cross-sectional area, (m\(^2\))

\( G \) = shear modulus, (N/m\(^2\))

\( I \) = second moment of area, (m\(^4\))

\( K \) = torsional constant, (m\(^4\)).

The K and C matrices are sparse therefore a large number of the calculations would entail multiplying by zero. If this could be avoided, a substantial reduction in execution times should occur. However, the form of K and C is not that of the normal banded matrix since the non-zero terms on a typical row are \( K_{i-i}, K_{i,i}, K_{i,i+1} \) and \( K_{i,N} \). Taking advantage of the symmetry property of normal stiffness matrices meant that a \( N \times N \) matrix could be reduced to a much smaller one. Special routines were written in order to manipulate matrices in this form.

The program has been modified to select its own time step with the intention of avoiding problems with convergence of the b series. The program also includes features which enables checks of energy input, energy dissipated, kinetic energy and potential energy to be made. These features enable the output of the program to be used in thermal analyses of a ball-screw system.

5. Comparison between simulated and measured results

The experimental set up was described by Pislaru [5] and consists in “opening” the position loop of the Beaver VC35 CNC machine tool axis drive. The input signal was introduced directly into the pre-amplifier of the analogue drive so as to give an input signal large enough to move the worktable more than 1 \( \mu \)m. The measured Bode diagrams [5] are presented in Figure 2. The input signal is white noise with 1.5 V maximum amplitude, 0.6 V standard deviation and 2 ms sampling time. The output signal is generated by the rotary encoder attached to the motor (solid trace) and the linear encoder (dotted trace).

The simulated results were generated by the C++ program describing the ball-screw dynamic behaviour in the case of open position loop for the Beaver vertical machining centre. The predicted frequency response for a sampling rate of 2 ms is shown on Figure 3. The solid line represents the response of the
rotary encoder attached to the motor and the dotted line represents the response of a linear encoder fitted on the machine bed, close to the slides which carry the saddle.

Analysing the measured and simulated results reveals:

(a) The amplitude plots for the motor encoder show a trough at about 90 Hz in both the predicted and measured cases.

(b) The simulated and measured amplitude plots for the linear encoder display a trough at around 50 Hz.

(c) The trough predicted at about 180 Hz is not as clear in the measured case because of the level of noise on the curve, although there is some indication of rising levels as 200 Hz is approached.

(d) A trough can be seen in the phase curves for the rotary encoder attached to the motor at about 75 Hz, with a broad "peak" following at just over 100 Hz.

(e) The simulated and measured phase diagrams for the linear encoder show troughs at 45 Hz and about 190 Hz.

6. Conclusion

The method of modelling the dynamic behaviour of a positioning system which includes a ball-screw with a moving mass has proved satisfactory when compared with dynamic data measured for the same stimuli conditions at the machine. In this way the proposed dynamic model of the ball-screw is validated.

The model, which can simulate a variety of non-linear behaviour in a ball-screw system, is subject to ongoing development. In addition to being able to model dynamic behaviour, it is hoped to use its energy dissipation features to help predict thermal behaviour.

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References

Figure 2 – Measured frequency response – white noise input
Figure 3 – Predicted frequency response – white noise input