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Curve fitting with arc splines for NC toolpath generation

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Abstract

Arc splines are important in automatically controlled complex curve cutting process. However, the problem of how to determine the parameter of arcs according to desired curve fitting accuracy has not been completely solved. This paper presents a new algorithm for finding arbitrarily close bi-arc splines. It is based on research on the characteristics of spiral curves. The algorithm has several features: (1) The curve fitting error can be calculated by a simple formula; (2) Number of segments can be predicted before curve fitting; (3) Compared with other methods, it is relatively simple and has a high computation efficiency.

1 Introduction

Many algorithms for approximately representing data by a G^0 curve made of straight-line segments have been published, because most modern CNC machine tools are capable of cutting circular arcs and straight lines. Arc splines, i.e. G^1 curves made of circular arcs and straight-line segments, are important. Research in this area has been active. Many papers are published on the G^1 arc spline approximation.

Sun and Zheng provide several choices of common tangent point of bi-arc approximation and their geometry explanation; Walton and Meek [1,2] have researched the relationship between conic NURBS, Bezier curves and Spirals, and have given a simple formula to estimate the approximation error. By this way, they realize bi-arc approximation of conic NURBS and Bezier curves; Ahn [4] utilize relationship between conic Bezier curves and spirals and provides the G^1 arc splines of conic Bezier curves. All those algorithms are for specific curves,

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but approximation of general curves still remains unsolved thoroughly.

For spiral curves, Marciniak and Putz [5] have proved that the mini-max approximation generates the lowest number of circular arcs for a spiral to satisfy a desired accuracy. However, the algorithm is not sufficient for real engineering application and the problem how to smoothly connect the circles at the joint of two spirals has not been solved; Qiu, Cheng and Li [5] improve the algorithm. The Continuous condition can be met and it can be applied to approximate general curves, but it is still not very practical because of its low calculation efficiency and the difficulty of selecting an initial value for an unknown variable.

The paper aims to develop a highly efficient and general curves bi-arc approximation algorithm. Based on the research on characteristics of spirals, it provides an optimal bi-arc approximation algorithm. A simple approximation error formula is given. To achieve high efficiency for a spiral curve, a near optimal bi-arc approximation is developed. The bi-arc approximation of general curves can be realized by subdividing it into spiral segments. The authors provide approximation examples for the above algorithm. These results confirm the validity of the algorithm developed by the authors.

2 Basic theory

The algorithm aims to bridge the gap between complex CAD models and the limited capabilities offered by conventional CNC machines. Because of the uncertain complexity of general curves, we believe that the approximation of general curves by a uniform curve is an ultimate method. To achieve the above goal, an approximation algorithm must: (1) process a general curve by a uniform means; (2) generate uniform data in a short time; (3) generate an appropriate volume of data. Although many papers have been published on the approximation of complex curves, few can meet the demands.

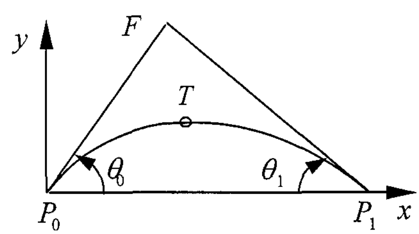

Figure 1: Bi-arc approximation of a curve

Definition 1: A bi-arc [1,2] is a curve that is made by joining two arcs in a G^1 manner.

Definition 2: A spiral [2] is a curve whose curvature is of one sign and monotone-increasing or monotone-decreasing as the curve is traversed.

Definition 3: A common tangent circular arc of a spiral curve is a circular, which consists of all common tangent points of all bi-arcs.

Definition 4: A left circular arc of a spiral is a circular arc which is tangent with the spiral at the start point and intersect at the end point of the spiral curve. A right circular arc of a spiral curve is a circular arc which is tangent at the end point of the spiral curve and intersect at the start point of the spiral curve.

Below theorems (all proofs are omitted) have been explored.

Theorem 1: The family of bi-arc of a spiral segment lies in the bi-arc region.

Bi-arc region is the region with boundary the left circular arc and the right circular arc.

Theorem 2: Optimal bi-arc approximation of a spiral curve satisfies the conditions: maximum approximation error is equal to minimum approximation error.

Theorem 3: Maximum error of bi-arc approximation of a spiral is not greater than

$$\left| l \cdot (tg \frac{\max(\theta_0, \theta_1)}{2} - tg \frac{\min(\theta_0, \theta_1)}{2}) \right| \tag{1}$$

Where l is the distance between the end-points of the spiral, θ_0, θ_1 are angle between tangent line of the spiral at the end-point and the line that joints end-points. Based on Theorem 1 and Theorem 2, an optimal bi-arc approximation algorithm has been developed. For a trade-off of processing time and size of data and based on Theorem 3, a more efficient approximate optimal bi-arc approximation algorithm is developed.

3 Optimal bi-arc approximation for a spiral curve

For a spiral AB, it is assumed that the approximation error at points P and Q are error extremum ϵ^+ and ϵ^-, and their coordinate are $(x(\phi_p), y(\phi_p))$ and $(x(\phi_q), y(\phi_q))$. The equation of the normal lines can be expressed as follows:

$$\left\{ \begin{aligned} y - y_a &= \frac{y'_a}{x'_a}(x - x_a) = k_a(x - x_a) \\ y - y(\phi_p) &= \frac{y'(\phi_p)}{x'(\phi_p)}(x - x(\phi_p)) = k(\phi_p)(x - x(\phi_p)) \end{aligned} \right\} \tag{2}$$

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To simply the expression, $(x(\phi_a),y(\phi_a))$ and $(x(\phi_b),y(\phi_b))$ are written as (x_a,y_a) and (x_b,y_b) .

Obviously, at an error extremum, the tangent lines on both the spiral and the circular arc are parallel to each other. Thus the normal lines of the spiral pass through the points A and B. In other word, the intersection point of both is the center point O_1 of the first circular arc.

$$\left\{\begin{aligned} x_{O1} &= \frac{k_a x_a - k(\phi_p)x(\phi_p) + y(\phi_p) - y_a}{k_a - k(\phi_p)} \\ y_{O1} &= \frac{k_a k(\phi_p)(x_a - x(\phi_p)) + k_a y(\phi_p) - k(\phi_p) y_a}{k_a - k(\phi_p)} \end{aligned}\right\} \tag{3}$$

For the same reason, the center point O_1 of the second circular arc can be expressed as follows:

$$\left\{\begin{aligned} x_{O2} &= \frac{k_b x_b - k(\phi_q)x(\phi_q) + y(\phi_q) - y_b}{k_b - k(\phi_q)} \\ y_{O2} &= \frac{k_b k(\phi_q)(x_b - x(\phi_q)) + k_b y(\phi_q) - k(\phi_q) y_b}{k_b - k(\phi_q)} \end{aligned}\right\} \tag{4}$$

According to Theorem 2,

$$|\varepsilon^+| = |\varepsilon^-| = E \tag{5}$$

$$\left\{\begin{aligned} L_p &= \sqrt{(x(\phi_p) - x_{O1})^2 + (y(\phi_p) - y_{O1})^2} = \rho_1 - m_1 m_2 E \\ L_q &= \sqrt{(x(\phi_q) - x_{O2})^2 + (y(\phi_q) - y_{O2})^2} = \rho_2 + m_1 m_2 E \end{aligned}\right\} \tag{6}$$

If the spiral has a positive radius of curvature then the value of m_1 is defined as 1, otherwise it is –1. If the radius of curvature of the spiral is monotonic increasing then the value of m_2 is 1□otherwise it is –1.

On the other hand, the first circular arc is tangent to the second circular arc. Thus,

$$|O_1 O_2| = |\rho_1 - \rho_2| \tag{7}$$

Thus,

$$\begin{cases} F_1(\phi_p,\phi_q) = L_p + L_q - \rho_1 - \rho_2 = 0 \\ F_2(\phi_p,\phi_q) = |O_1O_2| - |\rho_1 - \rho_2| = 0 \end{cases} \quad (8)$$

Where, $|O_1O_2| = \sqrt{(x_{O1} - x_{O2})^2 + (y_{O1} - y_{O2})^2}$,

$\rho_1 = \sqrt{(x_a - x_{O1})^2 + (y_a - y_{O1})^2}$, $\rho_2 = \sqrt{(x_b - x_{O2})^2 + (y_b - y_{O2})^2}$.

As (6) is a non-linear equation and has two unknown variables ϕ_a and ϕ_b. Using the Newton-Ralphon method and choosing suitable initial values, Equation (8) can be solved.

4 Near optimal bi-arc approximation of a spiral

Above an optimal approximation algorithm is presented, but it is not very practical because its low calculation efficiency. Besides, using the Newton-Ralphson method to solve non-linear equation, it is normally difficult to select an initial value for an unknown variable. A poor initial value may be cause no resolution. A new near optimal bi-arc approximation of a spiral curve has been developed. A spiral is divided into two segments by a point on it, and each of them is also a spiral. Based on Theorem 3, the near approximation errors are calculated. Adjust the position of the point until the absolute values of the near approximation errors are equal. The left circular arc of the first spiral curve and the right circular curve of the second spiral are connected as a new curve. The intersection of the common tangent circular arc of the spiral and the new curve are chosen as the common tangent point of the new bi-arc. The new bi-arc is a near optimal bi-arc.

5 Near optimal bi-arc approximation of a spiral within a given tolerance

In order to achieve desired accuracy, it is divided into segments according to the near approximation errors calculated by equation (1). A divide-and-conquer strategy is used to subdivide a spiral so that the maximum error is less than a given tolerance. Once the subdivision is achieved, a series of bi-arcs corresponding to adjacent segments are created. If the total number of segment is odd, the last bi-arc can be created according to the last segment.

Assume the curve is located in $[x_0,x_1]$, the given tolerance is E.

(1)Initial step > 0, usually step=x_1-x_0;

(2)According to equation (1), calculate approximation error e between current start point x and point x+step;

(3)If $|e-E|<\epsilon$, then generate a left circular arc and go to next, else adjust step, return (2);

(4)If x+step>x_1, go to next, else, set x+step as a new start point, return (2);
(5)Generate a series of bi-arcs.
For a general curve, it must be divided into spiral curves first.

6 Examples of approximation

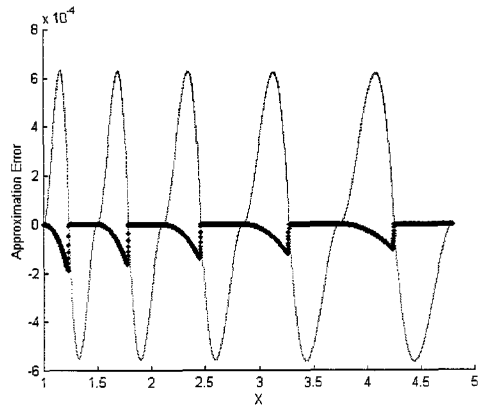

Figure 2: Approximation error of $y = x^2, x \in [1,5], e \le 0.001$

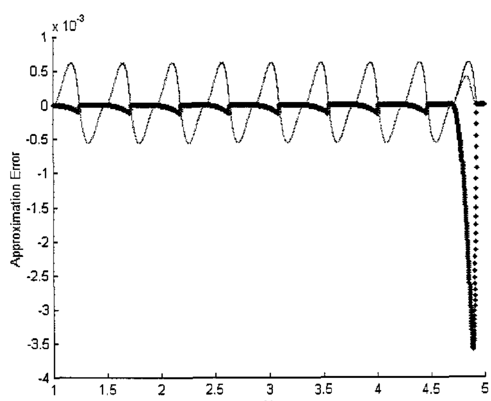

Figure 3: Approximation error of $y = e^x, x \in [1,5], e \le 0.001$

In order to verify our algorithm, we provide below examples. Curves $y = x^2, x \in [1,5]$ in Figure.2 and $y = e^x, x \in [1,5]$ in Figure.3 are both spirals. Figure.1 shows: (1) approximation error e of a single circular arc; (2) approximation error e' of near optimal bi-arc approximation; (3) Because e and e' are too close, we calculate errors e-e' and magnify them by 100. The error is shown by the thick black line. From Figure.2 and Figure.3, it can be seen e-e' is always oppositive to e and e'. In other word, new bi-arc approximation is better

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than the older single arc approximation. It proves that the algorithm in section 4 is effective.

In practice, many complex curves are not spirals. They must be converted into spiral segments and utilize above method.

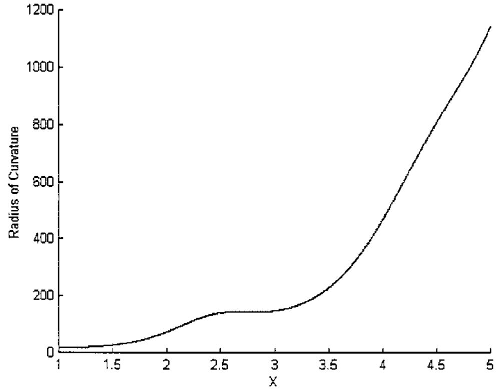

Figure 4: Curvature of $y = 4e^{x/2} + \sin(\pi x)/\pi^2, x \in [1,5], e \leq 0.001$

The basis of subdivision is the formula of curvature:

$$\rho = \frac{(1+(f'(x))^2)^{3/2}}{f''(x)} \tag{9}$$

According to the formula, we calculate the monotonic interval of the curve. The curve in Figure.4 is $y = 4e^{x/2} + \sin(\pi x)/\pi^2, x \in [1,5], e \leq 0.001$. It can be divided into four segments at x=1.1190,x=2.6460,x=2.8150. Figure.5 shows the approximation error.

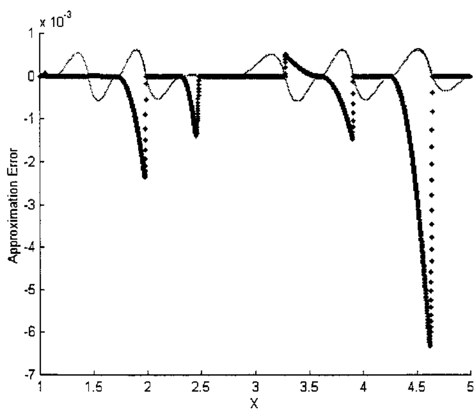

Figure.5 Approximation error of $y = 4e^{x/2} + \sin(\pi x)/\pi^2, x \in [1,5], e \leq 0.001$

7 Conclusion

The algorithm is simple and with high efficiency. Its programming is easy. We believe that to convert a general curve to spiral segments is an optimum method because it utilizes the curvature characteristic of it. The algorithm is very practical and useful.

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