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Solving Linear Differential Systems with Singular Points

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Abstract

Linear differential systems with isolated singular points may be solved in the vicinity of the singularity in terms of an asymptotic power series expansion of the solution. The package *SingularPoints* provides these solutions for vector systems of equations with the singularity located at the infinity point. The two common cases of regular singular point (*RSP*) and irregular singular point (*ISP*) cases are both included in the package. For the irregular singular point case the usual system with constant coefficients may also be solved exactly as a particular instance.

1 Introduction

The theory of Linear Ordinary Differential equations with singular points (see, e.g. Wasow [1],Coddington and Levinson [2]), represents a rich and classical field for which the symbolic computation of the solutions is a challenge for the capabilities of *Mathematica.* The present package is intented to be the natural complement to *DSolve* which already provides solutions for a linear homogeneous system of ODE with constant coefficients. We consider here the solutions for the vector system of homogeneous linear differential equation of the general form:

$$y' = M(x)y \tag{1}$$

where $M(x)$ is an n-order (complex-valued) matrix,and $y(x) = (y^{(1)}(x), ..., y^{(n)}(x))^t$, is the unknown (column) vector function. As it is well known, the space of solutions of the differential system (1) has dimension n. A set of n linearly independent

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solutions is called a fundamental set of solutions for Equation (1). The local point of view consider the fundamental solutions of (1) in the vicinity of some point, say $x = 0$. If $M(x)$ is analytic there, the solutions to Eq.(1) are also analytic in that neighborhood, and may be readily expressed as convergent power series.
The interest both in theory and applications is directed to the singular case, i.e., where $M(x)$ has some kind of (isolated) local singularity at a point, and we look for the analytic behaviour of the solutions in the vicinity of the singular point. The solutions may then cease to be analytic in the vicinity of the singularity. It is customary to locate the later at infinity by means of the obvious substitution $x- > 1/x$; the transformed differential system may be written in the general form:

$$y' = x^q A(x)y. \tag{2}$$

where q is an integer and the matrix $A(x)$ is now supossed to be developable as a convergent Laurent power series:

$$A(x) = A_0 + A_1/x + A_2/x^2 + ... \tag{3}$$

Equations of this type, are quite common in several areas such as fluids or plasma physics, where the matrix $A(x)$ may often depends on one ore more parameters, i.e., $A(x, k, ...)$; it is crucial in such applications to know the analytical behaviour as $x- > Infinity$ of the different "modes" corresponding to a set of independent solutions of Equation (2). The package *SingularPoint* may be used to find the exact asymptotic expansion of the solutions for the system of ordinary differential equations (2) with an isolated singularity at the infinity point.

2 Asymptotic solutions near the singularity: Regular and Irregular Singular Points

The classification of the singularity and the nature and structure of corresponding fundamental solutions is presented in the references [1,2]. It can be shown that two kinds of singularity may be enclosed by Equation (2):

i) Regular Singular Point (RSP).

ii) Irregular Singular Point (ISP).

These are respectively Equation (2) for $q = -1$ (RSP), and q a positive integer or zero (ISP). For systems of coupled differential equations with matrix $A(x)$ of order higher than two the task of finding a set of linearly independent asymptotic solutions of Equation (2) is generally recognized as a difficult problem; each independent solution may be generally written in terms of a product of a (potential or exponential) "leader" factor and an *asymptotic power series* (see e.g.,[3]) developable in the neighborhood of the singular point. Beside the linear scalar equation,

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and the (RSP) case, which are fairly easy to solve, the asymptotic solutions in the (ISP) case may include special kind of asymptotic series (*Puiseux series*), whose computation is included in the package.

The solutions generally depend on the spectral structure of the "leading" matrix A_0 (i.e., the first term in the series expansion of $A(x)$ in Equation (2)). The reduction to Jordan form for any matrix with repeated eigenvalues is then mandatory and the code for this is implemented as an auxiliary, thoroughly verified, package. If the eigenvalue spectrum of A_0 is moderately simple , the problem is solvable even if multiple eigenvalues are present.

The integer $q+1$ is the "singularity rank" for Equation (2). For $q=-1$ (rank of the singularity 0), we have the RSP differential equation:

$$y' = x^{-1}A(x)y \tag{4}$$

To solve this equation in the vicinity of the singularity, the method proposed by Wasow in reference [1] is used. The change of variable $y = TP(x)z$ is applied. T is the matrix that transforms A_0 (see Equation (3)),to the Jordan form,
i.e., $J_0 = T^{-1}\ A_0 T$, and $P(x) = I + P_1/x + P_2/x^2 + ...$ is a formal asymptotic series, chosen so as to transform the differential equation (4) into the simple form $z'(x) = x^{-1}\ J_0 z(x)$; this last has the exact solution, $Z(x) = Exp[J_0 Log(x)] = x^{J_0}$, where $Z(x)$ is a fundamental matrix.

This procedure is valid whereas the leading term in the Laurent expansion of $A(x)$, i.e., matrix A_0 in equation (3), has no eigenvalues with positive integer differences. The general structure of a fundamental matrix solution is then:

$$Y(x) = S(x)x^{J_0} \tag{5}$$

where $S(x) = S_0 + S_1/x + S_2/x^2 + ...$ may be computed using a recursive algorithm. If the spectrum of the matrix A_0 is formed by distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, then J_0 in Equation (5) is just the diagonal matrix:

$$J_0 = diag[\lambda_1, ..., \lambda_n] \tag{6}$$

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For the following simple equation:

$$y' = x^{-1}\begin{bmatrix} 0 & -1 \\ \frac{1}{x^2} & 0 \end{bmatrix} y$$

we may input the package function *RegularSingularPoint*$[Amatrix, \{i\}]$; the input arguments are the data matrix $A(x)$ and the order i of last term in the power series $S(x)$ to be computed. We obtain for the first fundamental solution (with $S(x)$ expanded up to the x^{-4} power):

$$y_1(x) = \begin{pmatrix} -1 - \frac{1}{64\,x^4} + \frac{1}{4\,x^2} \\ \frac{-1}{16\,x^4} + \frac{1}{2\,x^2} \end{pmatrix}$$

The series S(x) is not necessary a convergent series; it should be interpreted in the sense of an asymptotic series, where often the few first terms of the expansion adequately describes the behaviour as $x- > Infinity$.

For an Irregular Singular Point (q positive integer or zero), the solutions are often more complicated in nature. The general procedure begins again with a change of variable of the form $y = TP(x)z$, like the one used for the RSP case. A recursive algorithm is used to compute the succesive terms in the asymptotic series $P(x) = I + P_1/x + P_2/x^2 + ...$, in such a way that the transformed equation $x^{-q}z'(x) = J(x)z(x)$,will be block-diagonal,i.e., with $J(x)$ an x-dependent block-diagonal matrix corresponding to the Jordan blocks of J_0. If A_0 has n distinct eigenvalues, the original equation is de-coupled into n scalar equations which are readily solved. For $q = 0$ any particular "mode" solution corresponding to a particular eigenvalue, behaves at infinity as:

$$y(x) = x^g Exp(\lambda_j x)(u_0 + u_1/x + u_2/x^2 + ...) \tag{7}$$

with g a number, λ_j the eigenvalue and u_0 the corresponding eigenvector of A_0; if the singularity rank (the integer $q+1$ in Equation (2)) is greater than 1, the argument of the exponential factor is replaced by a polynomial of degree $q+1$ in x.

3 Irregular Singular Point with multiple eigenvalues

If the A_0 matrix in the expansion $A(x) = A_0 + A_1/x + A_2/x^2 + ...$ has multiple eigenvalues, and there not exists a complete set of eigenvectors, the previous procedure must be further extended, and a new kind of asymptotic expansion (*Puiseux series*) may rise. The code of the so-called *shearing transformations* is a necessary step to obtain the solution in this case (see the Wasow method in reference [1]). These transformations are linear change of variables defined by n-order non-singular matrices of the form $P(x) = diag[1, x^{-g}, x^{-2g}, ..., x^{-(n-1)g}]$ with a critical positive constant g to be computed. The purpose of this transformations is to decouple further a problem like Equation (2) when the leading matrix A_0 is nilpotent. The succesful coding of this step of the symbolic computation with the aid of *Mathematica* is a main characteristic of this package. The most general form of the ISP solution has the following structure of the fundamental matrix:

$$Y(x) = S(x)x^G Exp(Q(x)) \tag{8}$$

Here $Q(x)$ is a diagonal matrix whose diagonal elements are polynomials in $x^{1/p}$, p being a positive integer; G is a constant matrix and $S(x)$ is an asymptotic series wich may contain *fractional exponents*, i.e., integer powers in $x^{-1/p}$.
The package function $IrregularSingularPoint[Amatrix, \{q, i\}]$ is used with two arguments: $Amatrix$ corresponds to the data $A(x)$ matrix and the list $\{q,i\}$, inputs the q exponent in Equation (2) and the number of terms in the asymptotic series expansion to be computed. The next is an example from Wasow, which shows how asymptotic series with fractional exponents may rise if multiple eigenvalues are present; the equation to be solved is ($q = 0$):

$$y' = \begin{bmatrix} 0 & 1 \\ \frac{1}{(4x)} + \frac{5}{(16x^2)} & \frac{-2}{x} \end{bmatrix} y$$

and the two fundamental solutions that the package provides are:

$$y_1(x) = \begin{pmatrix} e^{-\sqrt{x} - \frac{3\log(2\sqrt{x})}{2}} \left(-2 - \frac{2}{\sqrt{x}}\right) \\ e^{-\sqrt{x} - \frac{3\log(2\sqrt{x})}{2}} \left(\frac{5}{2x^{\frac{3}{2}}} + \frac{5}{2x} + \frac{1}{\sqrt{x}}\right) \end{pmatrix}$$

and,

$$y_2(x) = \begin{pmatrix} e^{\sqrt{x} - \frac{3\log(2\sqrt{x})}{2}} \left(2 - \frac{2}{\sqrt{x}}\right) \\ e^{\sqrt{x} - \frac{3\log(2\sqrt{x})}{2}} \left(\frac{5}{2x^{\frac{3}{2}}} - \frac{5}{2x} + \frac{1}{\sqrt{x}}\right) \end{pmatrix}$$

In fact, these are exact solutions and the Puiseux asymptotic series are finished beyond the three first terms. With this package one is able, in principle, to solve any ISP differential system. To this end, the full machinery of matrix operations (the standard *Mathematica* package *LinearAlegebra'MatrixManipulation'*) is needed and used, along with a coded package for computing the Jordan form of square matrices with multiple eigenvalues. The power series with fractional exponents are automatically computed if necessary, thanks to the symbolic procedure centered around the concept of *shearing transformations* . These last include also exceptional cases of RSP computation when the leading term A_0 of the data matrix $A(x)$ expansion has eigenvalues with positive integer differences.

It must be noticed that a single higher-order differential equation may always be treated as a first-order system. A number of classical equations (Airy,Bessel, Legendre, Hypergeometric, etc.) may be therefore resolved by the package. The solutions that the package provides are always given in terms of a list with two matrices: the exact exponential leading factors in one of them, and the asymptotic series components of the solutions in the other. A fundamental matrix solution is then readily formed, to any degree of approximation in the asymptotic expansion.

The simplest example of ISP equation is of course an equation of the form $y' = Ay$, with A a constant matrix, i.e., just a system of linear homogeneous differential system with constant coefficients (in other words, all the terms, except the first one in the series expansion of $A(x)$ are all zero). As it should be the case, only the first term in the asymptotic series $S(x)$ survives so the vectorial "mode" solutions are just included in the equation (7) as a particular instance:

$$y(x) = Exp[\lambda x]u_0 \tag{9}$$

The usual constant coefficient system of ODE may be seen therefore just as the simplest case of an irregular singular point with $q = 0$. In the variable coefficient system, there appears a full asymptotic series with correcting terms as in Equation (7).

The most important case for which the package shows his full power is that of a higher order system. As a last example of an ISP equation the following is a four-order system with a parameter-dependent matrix $A(x, s)$:

$$y' = \begin{bmatrix} \frac{1}{(sx)} & 0 & 0 & 0 \\ \frac{-2}{(sx)} & 0 & 1 & \frac{-1}{x} \\ 0 & \frac{1}{x} & 0 & 0 \\ \frac{1}{(s^2 x)} & 1 & 0 & 1 - \frac{1}{(sx)} \end{bmatrix} y$$

We find for the leading matrix A_0 two eigenvalues: $(\lambda = 0, 1)$, but there does not exist a complete set of eigenvectors; the matrix cannot be reduced to diagonal form, so the Jordan form decomposition in blocks is the first step to simplify the problem. Finally, after a shearing transformation is done over the block corresponding to the multiple eigenvalue $(\lambda = 0)$, the package return the following fundamental solutions:

$$y_1(x) = \begin{pmatrix} 0 \\ e^{x + \frac{(-1-s)\ \log(x)}{s}} \left(\frac{-6 - \frac{2}{s}}{x^2} - \frac{1}{x} \right) \\ -\frac{e^{x + \frac{(-1-s)\ \log(x)}{s}}}{x^2} \\ e^{x + \frac{(-1-s)\ \log(x)}{s}} \left(1 + \frac{\frac{27}{2} + s^{-2} + \frac{15}{2\,s}}{x^2} + \frac{3 + \frac{1}{s}}{x} \right) \end{pmatrix}$$

$$y_2(x) = \begin{pmatrix} 0 \\ e^{-2\sqrt{x}+\frac{5\log(2\sqrt{x})}{2}} \left(\frac{59}{8192\,x^2} - \frac{1}{32\,s\,x^2} + \frac{151}{512\,x^{\frac{3}{2}}} + \frac{1}{2\,s\,x^{\frac{3}{2}}} - \frac{15}{16\,x} - \frac{1}{\sqrt{x}} \right) \\ e^{-2\sqrt{x}+\frac{5\log(2\sqrt{x})}{2}} \left(\frac{-303}{512\,x^2} - \frac{1}{2\,s\,x^2} + \frac{19}{16\,x^{\frac{3}{2}}} + \frac{1}{x} \right) \\ e^{-2\sqrt{x}+\frac{5\log(2\sqrt{x})}{2}} \left(\frac{-4427}{8192\,x^2} - \frac{17}{32\,s\,x^2} + \frac{265}{512\,x^{\frac{3}{2}}} + \frac{1}{2\,s\,x^{\frac{3}{2}}} - \frac{1}{16\,x} + \frac{1}{\sqrt{x}} \right) \end{pmatrix}$$

$$y_3(x) = \begin{pmatrix} 2^{\frac{2}{s}}\, x^{\frac{1}{s}} \\ 2^{\frac{2}{s}} \left(\frac{\frac{2}{s^4} - \frac{11}{s^3} + \frac{18}{s^2} - \frac{8}{s}}{x^2} + \frac{\frac{2}{s^2} - \frac{2}{s}}{x} \right) x^{\frac{1}{s}} \\ 2^{\frac{2}{s}} \left(\frac{\frac{2}{s^3} - \frac{7}{s^2} + \frac{4}{s}}{x^2} + \frac{2}{s\,x} \right) x^{\frac{1}{s}} \\ 2^{\frac{2}{s}} \left(\frac{\frac{-2}{s^4} + \frac{5}{s^3} - \frac{11}{s^2} + \frac{6}{s}}{x^2} + \frac{\frac{-3}{s^2} + \frac{2}{s}}{x} \right) x^{\frac{1}{s}} \end{pmatrix} \quad \text{and,}$$

$$y_4(x) = \begin{pmatrix} 0 \\ e^{2\sqrt{x}+\frac{5\log(2\sqrt{x})}{2}} \left(\frac{59}{8192\,x^2} - \frac{1}{32\,s\,x^2} - \frac{151}{512\,x^{\frac{3}{2}}} - \frac{1}{2\,s\,x^{\frac{3}{2}}} - \frac{15}{16\,x} + \frac{1}{\sqrt{x}} \right) \\ e^{2\sqrt{x}+\frac{5\log(2\sqrt{x})}{2}} \left(\frac{-303}{512\,x^2} - \frac{1}{2\,s\,x^2} - \frac{19}{16\,x^{\frac{3}{2}}} + \frac{1}{x} \right) \\ e^{2\sqrt{x}+\frac{5\log(2\sqrt{x})}{2}} \left(\frac{-4427}{8192\,x^2} - \frac{17}{32\,s\,x^2} - \frac{265}{512\,x^{\frac{3}{2}}} - \frac{1}{2\,s\,x^{\frac{3}{2}}} - \frac{1}{16\,x} - \frac{1}{\sqrt{x}} \right) \end{pmatrix}$$

All the terms up to the x^{-2} power for each asymptotic series are exactly computed. The behaviour at infinity of these solutions through the computed leading factors is also apparent. Similar examples of higher-order differential systems with singular points may be found in the field of plasma physics, where the usual task is to find out the particular mode solutions which are bounded at infinity (see Sanz[6]).

4 Conclusions

The package *SingularPoints* is written so as to resolve the general differential system (1). The code is developed following the general methodology to be found in Wasow [1]. Both the regular and irregular singularities are included. In particular,the general case of multiple eigenvalues in the leading matrix is implemented for the ISP equation. Beside this, the dimension of the system, and the rank of the singularity are arbitrarily chosen by the user. The solutions are obtained as asymptotic series expansions, and the the number of terms of these series are specified at the input. The only limitation is the possible algebraic complexity of the eigenvalue spectrum of the leading matrix of the system.

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