Curves and surfaces in the three dimensional sphere placed in the space of quaternions

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Abstract

In this article we will show how to use Mathematica in dealing with curves and surfaces in the three dimensional unit sphere $S^3$ embedded in the four dimensional Euclidian space $E^4$. Since $S^3$ is the Lie group of unit quaternions and at the same time it is a space of constant curvature, the analogy of the theory of curves in $E^3$ holds. We calculate curvature and torsion of curves in $S^3$ by Mathematica. The Gauss map $\nu$ of a surface in $E^4$ is decomposed into the two maps $\nu_+$ and $\nu_-$. If the surface is contained in $S^3$, we can define another Gauss map $\nu_s$. We use Mathematica to visualize the shapes of the images of these Gauss maps. Finally, the meaning of these images becomes clear through the notion of the slant surface.

1 Curves

The space $E^4$ is regarded as the space of quaternions with the multiplication

\[ ii = jj = kk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j \]

The unit sphere $S^3 \subset E^4$ is the Lie group of unit quaternions. Let $X_1 = (0,1,0,0), \quad X_2 = (0,0,1,0), \quad X_3 = (0,0,0,1)$ be the orthonormal frame of the tangent space $T_1S^3$ at $1 = (1,0,0,0)$ and $\tilde{X}_i$ be the left-invariant extension of $X_i$ over $S^3$. By $\nabla$, we denote the covariant differentiation
on $S^3$. In the geometry of $S^3$, left translations play the analogous role to that of parallel displacements in the ordinary Euclidean space $\mathbb{E}^3$ (cf. Chen&Tazawa[3]). The unit tangent vector of a curve $c(s)$ parametrized by arc-length $s$ is expressed as

$$t(s) = \frac{d}{ds}c(s) = \sum_{i=1}^{3} f_i(s) \tilde{X}_i(c(s))$$

The curvature, unit normal vector, unit binormal vector, and torsion are defined respectively by

$$\kappa(s) = \| \nabla_{t(s)} t \|, \quad n(s) = \frac{1}{\kappa(s)} \nabla_{t(s)} t,$$

$$b(s) = t(s) \times n(s), \quad \tau(s) = - \langle \nabla_{t(s)} b, n(s) \rangle$$

**Example 1**

$$c(t) = \left( \frac{1}{\sqrt{2}} \right) \left( \sin(\sin(t)), \cos(\sin(t)), \sin(t^2 - 4t), \cos(t^2 - 4t) \right)$$

The curvature and torsion are calculated and plotted by Mathematica in Figure 1.

![Figure 1: curvature and torsion](image)

**Example 2**: Geodesic

Let $a$ and $b$ be any constants. A geodesic (great circle) is given by

$$c_{ab}(t) = (\cos t, -\sin t \cos b \sin a, \sin t \cos b \cos a, \sin t \sin b)$$

The curvature of a geodesic vanishes.
Example 3: Helix in $S^3$

For any constant $a$ and $b$, the curve defined by

$$h_{ab}(t) = \left( a \sin t, a \cos t, \sqrt{1 - a^2 \sin^2 bt}, \sqrt{1 - a^2 \cos^2 bt} \right)$$

has constant curvature and torsion. Moreover, the angle between its tangent vector and the left invariant vector field $\tilde{X}_1$ is constant. We call this curve a helix in $S^3$ (cf. Chen&Tazawa[3]). The projections to the three dimensional coordinate spaces of a helix in $S^3$ with $a = 1/64$ and $b = 1/512$ are shown in Figure 2.

![Figure 2: projections of a helix](image)

2 Gauss map

The exterior algebra $\Lambda^2 E^4$ of $E^4$ is decomposed into two eigen spaces $\Lambda^2_+ E^4$, $\Lambda^2_- E^4$ of the Hodge's star operator

$$* : \Lambda^2 E^4 \rightarrow \Lambda^2 E^4$$
corresponding to the eigen values $\pm 1$, respectively. Let $S^2_+, S^2_-$ be the spheres of radius $1/\sqrt{2}$ in $\Lambda^2_+E^4$, $\Lambda^2_-E^4$ centered at the origin. Then the Grassmannian manifold $G(2, 4)$ of all oriented 2-planes in $E^4$ is identified with the Riemannian product

$$G(2, 4) \equiv S^2_+ \times S^2_-$$

of these spheres through the identification $V \equiv e_1 \wedge e_2$, where $\{e_1, e_2\}$ is an orthonormal frame of a plane $V \in G(2, 4)$.

For a surface $f : D \subset E^2 \to E^4$, the Gauss map $\nu : D \to \Lambda^2 E^4$ takes its value to be the tangent plane at each point of the surface. It is decomposed into the two maps (cf. Chen&Tazawa[2]).

$$\nu_+ : D \to S^2_+$$

$$\nu_- : D \to S^2_-$$

**Example 4**

$$(u, v) \to (u, u^2 - 3v, uv, u - v^3)$$

The two Gauss images of this surface are plotted in Figure 3.

**Example 5**

A surface $z \to (z, z^2)$, identifying $C^2 \equiv E^4$. The Gauss images are shown in Figure 4. Although the image of $\nu_+$ cannot be seen in the figure, it is a singleton on the axis of the first coordinate.
Example 6: Flat torus

\[(u, v) \rightarrow \frac{1}{\sqrt{2}} (\cos u, \sin u, \cos v, \sin v)\]

The images of \(\nu_+\) and \(\nu_-\) in Figure 5 are the great circles perpendicular to the axis of the first coordinate.

Figure 5: Gauss images of a flat torus
3 Another Gauss map

If a surface is contained in the 3-sphere, \( f : D \subset \mathbb{E}^2 \rightarrow S^3 \subset \mathbb{E}^4 \), we can define another Gauss map as follows (cf. Chen&Tazawa[3]). Let \( n : D \rightarrow T_f(u,v)S^3 \) be the unit normal vector field in \( S^3 \) along the surface. Then, the spherical Gauss map \( \nu_s : D \rightarrow S^2 \subset T_1S^3 \) is defined by \( \nu_s = L_{n(x)}^{-1} \circ n \), that is, \( \nu_s(u,v) = L_{f(u,v)}^{-1}(n(u,v)) \). Since the left translation in \( S^3 \) is analogous to the parallel displacement in \( \mathbb{E}^3 \), the spherical Gauss map is a natural generalization of the ordinary Gauss map.

Example 7

The image under the spherical Gauss map of the flat torus given in Example 6 is plotted in Figure 6.

![Figure 6: \( \nu_s \)-image of a flat torus](image)

4 Slant surfaces

A surface in the 4-dimensional space can also be regarded as a surface in the 2-dimensional complex space \( f : D \subset \mathbb{E}^2 \rightarrow \mathbb{C}^2 \equiv \mathbb{E}^4 \). We denote by \( J \) the complex structure of \( \mathbb{C}^2 \). A surface is called an \( \alpha \)-slant surface, if the angle between \( JX \) and \( T_pM \) is constant, i.e., \( \angle (JX, T_pM) = \alpha \) for any \( X \in T_pM \) and any \( p \in M = f(D) \) (cf. Chen[1]).

Let \( \{e_1, e_2, e_3, e_4\} \) be the canonical basis of \( E^4 \). Then, the following three are equivalent (cf. Chen&Tazawa[3]).

1. \( f \) is an \( \alpha \)-slant surface.
2. The angle between \( \nu_+(u,v) \) and \( \frac{1}{\sqrt{2}} (e_1 \wedge e_2 + e_3 \wedge e_4) \) is \( \alpha \).
3. The angle between \( \nu_s(u,v) \) and \( X_1 \) is \( \alpha \).

A family of slant surfaces contained in the 3-sphere is obtained by the quaternionic product of a geodesic in \( S^3 \) and a helix in \( S^3 \) (cf. Chen&Tazawa[2]).
Example 8

We choose a helix $h(s)$ in $S^3$ parametrized by arc-length $s$ so that its coefficient of the unit tangent vector satisfies $f_1(s) = -\sin(\pi/10)$. We obtain such $h(s)$ by solving a system of ordinary differential equations making use of NDSolve. We also choose a geodesic $\gamma(t) = c_{ab}(t)$ in Example 2 with $a = \pi/10$ and $b = \pi/4$. Define a map by $\varphi(x, y, z, w) = (x, y, w, z)$ and put $f(s, t) = \varphi(\gamma(t) \cdot h(s))$, where $\cdot$ is the quaternionic product. The projections of this surface to the coordinate spaces are shown in Figure 7.

The surface is a $\pi/10$-slant surface and its image under $\nu_+$ is contained in a circle perpendicular to the first coordinate axis as is seen in Figure 8.
As can be seen from the images under $\nu_+$, the surface of Example 4 is not a slant surface. The surface of Example 5 is a 0-slant surface (i.e., a holomorphic surface) and the flat torus in Example 6 is an $\pi/2$-slant surface (i.e., a totally real surface) (cf. Chen & Tazawa [2], Chen & Tazawa [3], Tazawa [5]).

References

1. Paper in a journal:


2. Edited book: