Fuzzy reasoning in irrigation network design
L.S. Vamvakeridou-Lyroudia
E. & L. Lyroudias, Consulting Engineers Ltd, 122 Tatoiou Street, 14671 Athens, Greece

Abstract
In this paper, fuzzy reasoning is used for the optimal design of branched pressure water distribution networks under multiple loadings. Non-linear fuzzy membership functions are introduced to velocity and pressure constraints. Optimization is carried out by an extension of the classical Dynamic Programming scheme for networks. An original mathematical approach to the problem is presented. There is also a discussion on the selection of partial membership functions and aggregation operators.

1. Introduction
In classical design optimization models, any network is designed according to the "worst" or "heavier" loading, that assumes maximum (estimated) consumption at the nodes. Optimal solution is selected as the "best solution" in the set of feasible solutions, namely the solution that minimizes cost, given a set of constraints. For any solution, feasibility is assumed simply by comparing an arithmetic value to a lower or upper bound, that has been previously rigidly set by the engineer. Should deviation by even the smallest significant real number occur, the solution is automatically rejected as non-feasible. This is the "crisp logic" classical approach to the problem (Walters & Cembrowicz [6]).

Even if multiple loadings are used as network design parameters, feasibility is estimated by Boolean two-valued reasoning, that is, given any system state there are only two possible answers: Yes (accepted) or No (not accepted).

Engineers know by experience, that there is seldom a definite answer as to whether a system state is acceptable or not. This is due to many facts: (a) definition of the "worst" loading is based on probabilistic parameters (b) network mathematical simulation implies assumptions (e.g. as to friction head losses, or lumped node consumptions) and (c) small deviations from constraint upper and lower arithmetic bounds do
not cause system failure. Thus, fuzziness is introduced to network design. A linear
fuzzy programming model was presented by Goulter and Bouchart [2], but no further
similar development exists (Goulter [1]).

In this paper an original mathematical model for network design is presented,
that emphasizes on acceptability of solutions using fuzzy reasoning. It can be applied
to branched networks with multiple loadings. Layout is considered fixed, and con-
sumptions lumped at nodes. Thus the model can be directly applied to irrigation net-
work design. Optimization is carried out by using an extension of the classical
Dynamic Programming model by the same author (Vamvakeridou-Lyroudia [5]).

2. Fuzzy sets and operators

A fuzzy set is a set with "fuzzy" or not clear boundaries. For instance a possible fuzzy
set is "the set of tall people". Some people clearly belong to this set, some others, cer-
tainly, do not. In any case there is a transition zone, where judgement is not clear.
Mathematically, fuzzy sets are defined as follows:

Let $X$ be the universal set. A fuzzy set $A$ in $X$ is defined by its membership
function $\mu_A(x)$, which denotes the membership of each $x \in X$ in the fuzzy set $A$.
Usually the range of values for membership functions is between 0 and 1, inclusive. So,
the nearer the value of the membership function to 1, the higher the grade of
membership of $x$ to $A$, and vice versa.

Operators are defined on fuzzy sets in much the same way as on crisp sets. If
$A$ and $B$ are fuzzy sets, fuzzy union $\mu_{A \cup B}(x)$ is defined as an operand $\max[\mu_A(x), \mu_B(x)]$ and
respectively fuzzy intersection $\mu_{A \cap B}(x)$ is defined as an operand $\min[\mu_A(x), \mu_B(x)]$. Usually
(but not necessarily) fuzzy union and fuzzy intersection are defined as

$$\mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)] \quad (1)$$

$$\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)] \quad (2)$$

In this case they are respectively called the max and min operators.

When several fuzzy sets are combined to produce a single set, aggregation
operations are employed. Aggregation operators are applied to $n$ ($n \geq 2$) fuzzy sets
$A_1,A_2,\ldots,A_n$ to combine them to a single set $A$ (Klir & Folger [4]). Any aggregation op-
erator is defined as

$$\mu_A(x) = h[\mu_{A_1}(x), \mu_{A_2}(x), \ldots, \mu_{A_n}(x)] \quad (3)$$

Fuzzy union and intersection are aggregation operators. Apart from these,
generalized means may be used as aggregation operators, defined as

$$h_\alpha = \left(\frac{\mu_{A_1}(x)^\alpha + \mu_{A_2}(x)^\alpha + \cdots + \mu_{A_n}(x)^\alpha}{n}\right)^{\frac{1}{\alpha}} \quad (4)$$

where $\alpha$ ($\alpha \in \mathbb{R}, \alpha \neq 0$) is a shape parameter by which different means are distin-
guished. The arithmetic mean $h_1$ is obtained for $\alpha=1$, the geometric mean $h_0$ for $\alpha=0$,
while the harmonic mean $h_{-1}$ is obtained for $\alpha=-1$. Moreover, aggregators $h_\alpha$ can be
generalized into weighted means, by the use of weighting coefficients \( w_i \geq 0, i=1, \ldots n \) that express the relative importance of the aggregated sets

\[
h_{\alpha,W} = \left( \frac{1}{\alpha} \sum_{i=1}^{n} w_i \mu_{A_i} \right) \sum_{i=1}^{n} w_i = 1
\]

3. Fuzzy reasoning in network design

In the case of network design optimization by Dynamic Programming (DP), stages are the links, determined by consecutive nodes. Decision variables are the diameters at the links, and

(a) Alternatives and decision space are determined by diameter, pressure and velocity constraints,

(b) States are the energy heads at the nodes,

(c) State equations are formed by the friction energy losses at the links, so that states at the nodes are related to each other, and

(d) Objective function is the minimization of the total cost, which is obtained by adding the costs (return functions) of all previous stages.

Generally speaking, fuzziness (in terms of uncertainty or vagueness) may be introduced in any or all of those four components, depending on the nature of the system. In the model presented here, fuzziness is introduced in constraints and the objective function. Fuzziness in friction coefficients is ignored, whereas fuzziness in rate of flow computations (or demands at the nodes) is taken into account by assuming \( M \) multiple loadings. In network design there are three groups of constraints:

(a) Diameter constraints, because decisions must comply with the set of commercial diameters. This set is definitely crisp, so diameter constraints remain crisp constraints for all stages, as in classical DP.

(b) Pressure constraints. Fuzziness may certainly be introduced in pressure constraints at all nodes, because there is no clear distinction between what is acceptable or not (e.g. an alternative need not be rejected if pressure at the node is 0.1 m less than the minimum required). So the pressure occurring at each node is related to the set of feasible solutions by a membership function \( \mu_p \). It should be noted that pressure at each node directly depends on the state variable values (energy heads at each node), so that the value of this function depends on all previous stage decisions.

(c) Velocity constraints, for specific rate of flow at the links. Fuzziness may be introduced in velocity constraints in the same way as in pressure constraints. Velocity at each link is related to the set of feasible solutions by a membership function \( \mu_v \). In the case of velocity constraints, \( \mu_v \) values depend on the actual stage only, because they are related to the decision and not the state variables. Naturally, for each stage and loading, there is a distinct \( \mu_v \) value.
In classical network design, the goals are expressed as cost minimization. By using DP for each state and stage the minimal cost is selected by ordering alternatives in terms of total cost. If fuzzy decision is applied, the set of minimal costs for each stage and state becomes fuzzy, because fuzziness is already associated with the alternatives. A proper aggregation operator $h$ can then be selected to simulate any necessary trade-off, preference or compromise that exists between goals and constraints. The objective function is then transformed to maximizing the system total membership function, which is equivalent to minimizing fuzzy aggregated costs.

4. Network optimization model

If DP is applied to network design, usually Backwards Dynamic Programming (BDP) is applied, so that computations start from the last stage. The network consists of $NP$ links and $NJ=NP+1$ nodes. Each link connects two consecutive nodes and forms the computational element of the model. Stages $i$, $i=1, \ldots, N$ of the model are the links $p$, $p=1, \ldots, NP$, so that the total number of stages $N$ is equal to the number of links $NP$. Stages are numbered starting at the most upstream node and proceeding gradually downstream, so that the number identification of each stage is greater than all upstream links (and stages).

It is assumed that the network operates under $M$ loadings. Therefore for each stage there are $M$ rate of flow values $Q(i,m)$, $i=1, \ldots, N$, $m=1, \ldots, M$.

The crisp set for possible diameters is the set $DPOS$ of commercial diameters $D$ available, arranged in ascending order

$$DPOS(i, k_j) \quad i=1, \ldots, N \quad k_j=1, \ldots, NK(i) \quad (6)$$

where $k_j=1$ stands for the smallest diameter available for stage $i$ and $k_j=NK(i)$ respectively for the largest one. Accordingly, the set of possible velocities $VPOS(i,k_j,m)$ may be formed

$$VPOS(i, k_j, m) \quad i=1, \ldots, N \quad k_j=1, \ldots, NK(i) \quad m=1, \ldots, M \quad (7)$$

Velocity constraints then, are written

$$V_{\min}(D) \leq VPOS(i, k_j, m) \leq V_{\max}(D) \quad (8)$$

In classical programming, should any $VPOS(i, k_j, m)$ exceed the limits, $DPOS(i,k_j)$ would be rejected and excluded from the decision space. Using fuzzy sets, any $VPOS(i, k_j, m)$ belongs to the set of feasible alternatives by the following membership functions $\mu_{VM}(i, k_j, m)$

$$\mu_{VM}(i, k_j, m) = \begin{cases} 1 & \text{if } V_{\min}(D) \leq VPOS(i, k_j, m) \leq V_{\max}(D) \\ (1 + X_1 p^1)^{-1} & \text{if } V_{\min}(D) > VPOS(i,k_j,m) \\ (1 + X_2 p^2)^{-1} & \text{if } V_{\max}(D) < VPOS(i,k_j,m) \end{cases} \quad (9)$$
where

\[ X_1 = \frac{|V_{POS}(i,k,m) - V_{min}(D)|}{V_{min}(D)} \]

and \( p_1, p_2 \) shape parameters and \( 0 \leq X_1 < 1 \). There is no upper limit for \( X_2 \). Theoretically \( 0 \leq X_2 < \infty \), but the membership function value is close to zero for large deviations. It should be noted that membership functions are not linear. Previous similar models for networks (Goulter and Bouchart [2]) and constraint deficiencies in water resources (Kindler [3]) assumed linear membership functions and defined the support of the fuzzy set by a crisp lower bound, that represented the maximum acceptable deficiency with membership function equal to 0. Membership function decreased linearly from 1 (upper desired constraint value) to 0 (lower bound). In reality membership functions represent the level of satisfaction from each alternative, which does not alter linearly. The shape parameters \( p_1 \) and \( p_2 \) are introduced in order to simulate dissatisfaction. If \( p_1 \neq p_2 \) deviation tolerance is different for the max or min bound. Also, in this way, maximum acceptable deviation is not crisply set, but the membership function asymptotically reaches 0 for large values. Recommended values for the shape parameters are \( 2 < p_1, p_2 < 5 \).

By eqn[9] \( M \) membership functions are computed for each alternative \((i,k)\) that correspond to \( M \) loadings. An aggregator is used for the total membership function \( \mu_{\nu}(i,k) \) in the feasible velocities set

\[ \mu_{\nu}(i,k) = h_{\nu} \left[ \mu_{\nu M}(i,k,1), \ldots, \mu_{\nu M}(i,k,M) \right] \]  

(11)

for \( i=1, \ldots, N, k=1, \ldots, KN(i) \), while \( h_{\nu} \) is the velocity aggregation operator. Operator selection may vary from the min operator (fuzzy intersection- stronger) (eqn[2]) to any generalized weighted mean (weaker) (eqn[5]), in case of assigning different weights (or importance) to the \( M \) loadings.

The array of velocity membership functions \( \mu_{\nu}(i,k) \) is computed once, prior to the actual DP process, because it is not affected by state variables. In the same way, two additional arrays are formed: The array of possible head losses, by using e.g. the Darcy-Weisbach formula

\[ DH_{POS}(i,k,m), i=1, \ldots, N, k=1, \ldots, NK(i), m=1, \ldots, M \]  

(12)

and the array of possible costs for all alternatives

\[ C_{POS}(i,k), i=1, \ldots, N, k=1, \ldots, NK(i) \]  

(13)

Decision variables are the diameters of the links \( D(i,k) \), or rather the values \( k, \) that form the alternative space for each stage. States are the values of the piezometric head \( (H, k)_m \) at the stage upstream node \( i, i=1, \ldots, N, m=1, \ldots, M \). Feasible state values \( (H, k) \) must comply to pressure constraints

\[ H_i - Z(i) \geq P_{\text{min}}(i) \]  

(14)
where \( Z(i) \) is the node elevation (in m) and \( P_{\text{min}}(i) \), the minimum pressure (in m) desired at the node. Using membership functions in the same way as with velocity constraints, for any alternative \( k_i \), loading \( m \) and stage \( i \), the membership function of the state \((H_i, k_i)\) in the set of feasible solutions is:

\[
\begin{align*}
\mu_{pm}(H_i, k_i) &= 1 \quad \text{if } P_{\text{min}}(i) \leq H_i - Z(i), \\
&= 0.5 + 0.5 \left[ 1 - \left( \frac{X}{X_{05}} \right) \right]^\frac{1}{p3} \quad 0 < X \leq X_{05} \\
&= 0.5 \left[ 1 - \left( \frac{X - X_{05}}{1 - X_{05}} \right) \right]^\frac{1}{p4} \quad X_{05} < X \leq 1
\end{align*}
\]

where \( p3, p4, X_{05} \) shape parameters and

\[
X = \frac{|H_i - Z(i) - P_{\text{min}}(i)|}{P_{\text{min}}(i)}
\]

Membership function consists of a concave (for small deviations) and a convex elliptic part (for larger deviations). Inflection point is set for \( X = X_{05} \), when \( \mu_{pm} = 0.5 \).

In the case of fuzzy logic, no discrete dynamic programming may be applied. State variables are obtained by using a computational step \( H_{\text{STEP}} \) at the nodes (e.g. 0.1 m).

Let \((H_i, k_i)\) be a possible state value at node \((i)\). For stage \((i)\) there are \( k_i = 1, \ldots, K_N(i) \) diameter (decision value) alternatives. Each alternative represents a stage velocity membership value \( \mu_v(i, k_i) \) (eqn. [11]). There exists also a stage pressure membership function value \( \mu_p(H_i, k_i) \) which is estimated, with the help of state functions (eqn. [17]) as follows:

\[
(H_{i+1}, k_i) = (H_i, k_i) - DHPOS(i, k_i, m)
\]

\[
\mu_p(H_i, k_i) = h_p[\mu_{pM}(H_i, k_i), \mu_p(H_{i+1}, k_i)_{m=1}, \ldots, \mu_p(H_{i+1}, k_i)_{m=M}]
\]

where \( h_p \) is the stage pressure aggregation operator. The component \( \mu_{pM}(H_i, k_i) \) stands for the membership function of the \( H_i \) value at node \( i \). In case \( i+1 \) is a downstream end, there are no downstream conditions to take into account. If \( S_N \) downstream junctions start from node \( i+1 \), then downstream conditions are simulated as follows:

\[
\mu_p(H_{i+1}, k_i)_{m} = h_D[\mu_{PM}(H_{i+1}, k_i)_{m}, \mu_p(H_{i+1}, k_{i+1})_{m,s_1}, \ldots, \mu_p(H_{i+1}, k_{i+1})_{m,s_N}]
\]

where \( h_D \) stands for the downstream aggregator and \( s_1, \ldots, s_N \) are the downstream branches.
As with the velocity aggregator, $h_p$ and $h_b$ operator selection may vary from the min operator (fuzzy intersection - stronger) (eqn[2]) to any generalized weighted mean (weaker) (eqn[5]), in case of assigning different weights (or importance) to the M loadings or the NS downstream junctions. So the total combined membership function $\mu_{pV}(H, k_i)$ for both velocity and pressure constraints is

$$\mu_{pV}(H, k_i) = h_{pV}[\mu_p(H_i, k_i), \mu_V(i, k_i)]$$ (20)

where $h_{pV}$ stands for the stage total aggregator. The return function for each stage is

$$C(i, k_i) = \frac{CPOS(i, k_i)}{\mu_{pV}(H_i, k_i)}$$ (21)

so the total objective function becomes

$$F^*_i(H_i, k_i) = \min \left[ C(i, k_i) + \sum_{s=1, t \in s}^{NS} F^*_{i+t}(H_{i+t}, k_{i+t}) \right]$$ (22)

for $i=1, \ldots, N$, $t=i+1, \ldots, N$, where $s$ stands for each junction and the sum $\Sigma$ represents partial optimal return functions taken cumulatively into account. The DP algorithm starts downstream and proceeds gradually upstream, stage by stage. When the most upstream stage is reached, the optimal $k_i$ values, that correspond to the optimal diameters are retrieved by forward tracing the stages.

In case of gravity fed networks, the system is complete. In case of pumping stations at the head, the pumping station is simulated as another (last) stage, its cost added to the cumulative return functions, as in crisp DP.

There is one important point to add. By eqn[21] constraint violation is taken into account as penalty that augments the cost. So the objective function in eqn[22] is equivalent to the following:

$$\max \left[ \sum_{i=1}^{N} \frac{CPOS(i, k_i)}{\mu_{pV}(H_i, k_i)} \right]$$

$$\frac{CPOS(1, k_1)}{\mu_{pV}(H_1, k_1)} + \cdots + \frac{CPOS(N, k_N)}{\mu_{pV}(H_N, k_N)}$$ (23)

Eqn. [23] stands for the weighted harmonic mean fuzzy aggregator of the system.

5. Remarks and conclusion

The mathematical model for branched network optimization using fuzzy reasoning has been implemented as the software package IRRIGFUZ, which is an extension of the IRRIGOPT program presented by the same author in Hydrosoft/90 [5] and initially in Hydrosoft/86. The fuzzy DP algorithm was written in Fortran 5.0, running under DOS on PCs, with 80386 or 80486 processor.
At this stage, membership function shape parameters and aggregators are selected by the user, with the help of a menu. The model is now being experimentally applied to various real networks, in order to determine the effects of parameter selection to the model, so as to establish guidelines for non-specialized users. So far, "best" results have been obtained with the following:

\[ \mu_{\text{VM}}; p_1=4, p_2=4, \mu_{\text{PM}}; p_3=2, p_4=1/p_3, X05=0.2 \]

- \( h_{\text{vi}} \): min operator or weighted arithmetic mean
- \( h_{\text{pi}} \): min operator or weighted geometric mean
- \( h_{\text{v}} \), \( h_{\text{p}} \): min operator

Thus, up to 15% cost reduction has been achieved, with maximum constraint deviation 20% in less than 5% of the nodes and links. Pressure constraint deviation seems to be critical for high head values, while velocity constraint deviation is critical for low head values.

6. References


