Solving initial-boundary heat conduction problems using the multistage–multigroup least squares method
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Abstract

The paper presents a new approach of solving the initial-boundary heat conduction problems containing data subject to uncertainties. When the mathematical model contains measurement results, using the adjustment procedure, we can verify its accuracy and determine the influence of the measurement errors on the solution. The adjustment procedure is very often based upon the least squares principle. The constraint equations are usually derived using numerical methods. The coefficient matrix of the set of constraint equations is then a sparse matrix. We exploited the structure of this matrix and derived the multistage–multigroup method (MM method). The method was tested on the estimation of unsteady-state temperature fields in boiler shells. Both overdetermined and ill-posed cases are considered.

1 Introduction

In many areas of engineering research the need of more accurate modeling of physical processes is vital. A frequent situation is that a check on correctness of the model is carried out by a comparison of the calculation results with the data. The most appropriate way of ensuring the best matching of a model to data subject to errors, is to estimate the unknown parameters of the model from data. This is possible using the adjustment
24 Advanced Computational Methods in Heat Transfer

There are many advantages to be gained from this approach. We can verify the accuracy of the model, achieve more probable calculation results, and determine the influence of uncertainties of data on them. The adjustment procedure is usually based on the least squares criterion \[1\]. The constraint equations are frequently derived using numerical methods. The system of these equations can be written as

\[
A \mathbf{v} + B \mathbf{y} = \mathbf{w}
\]

where

\[
\mathbf{v} = \mathbf{\alpha} - \mathbf{1}, \quad \mathbf{y} = \mathbf{\beta} - \mathbf{x}, \quad \mathbf{w} = \mathbf{c} - A \mathbf{l} - B \mathbf{x}
\]

\(A, B\) denote \(r \times s\) and \(r \times n\) matrices of coefficients, respectively, \(c\) is \(r \times 1\) vector of free terms, \(l\) is \(s \times 1\) vector of measurement results, \(v\) is \(s \times 1\) vector of the corrections to the measurement results, \(x\) is \(n \times 1\) vector of unknowns, \(y\) is \(n \times 1\) vector of corrections to unknowns, and \(\alpha\) and \(\beta\) denote the estimates of measured quantities and unknowns, respectively. The vectors \(\mathbf{v}\) and \(\mathbf{y}\) estimated in accordance with the least squares criterion can be calculated only if the ranks of the matrices \(A\) and \(B\) are equal to \(r\) and \(n\), respectively. Unfortunately, these conditions in models of temperature fields are not fulfilled at all. The solution could be obtained by eliminating some or all of the unknowns from the set of constraint equations but this is difficult to perform, especially when the transient problems are considered. To effect the solution with avoidance of this problem the unified least squares method can be employed \[2\], \[4\].

2 The unified least squares method

The fundamental approach to the unified least squares method lies in the treatment of all variables taken into adjustment procedure as observations. This means that for the unknowns we have to estimate their preliminary values and uncertainties. Frequently these variables can be evaluated on the basis of their physically feasible range. Noting \(A = A_l, B = A_y\) the set of constraint equations takes form:

\[
\bar{A} \bar{v} = \mathbf{w}
\]

where

\[
\bar{A} = [A_l, A_y], \quad \bar{v} = [\mathbf{v}^T, \mathbf{y}^T]^T, \quad \bar{l} = [\mathbf{l}^T, \mathbf{x}^T]^T
\]
When transient problems are considered the number of constraint equations can be extremely large. This is valid especially when the unsteady temperature fields in the elements of technological systems are determined, because the number of constraint equation is a product of the number of difference equations and the number of total time steps. In such situations, the multistage method can be used [3]. In the multistage approach instead of the system (3) the subsystems (5) are considered:

\[ \mathbf{A}_i \mathbf{v} = \mathbf{w}_i, \quad i = 1, 2, \ldots, m \]  

(5)

and \( \mathbf{A}_i, \mathbf{w}_i \) are respectively \( r_i \times s, \quad r_i \times 1 \) matrices. Matrix \( \mathbf{A} \) and vector \( \mathbf{w} \) take the forms:

\[ \mathbf{A} = [\mathbf{A}_1^T, \mathbf{A}_2^T, \ldots, \mathbf{A}_m^T]^T; \quad \mathbf{w} = [\mathbf{w}_1^T, \mathbf{w}_2^T, \ldots, \mathbf{w}_m^T]^T. \]  

(6)

In mathematical models of unsteady temperature fields the coefficient matrix of the set of constraint equations is a sparse matrix. The efficiency of solving each large sparse matrix problem is significantly affected by storing and operating on nonzero elements only. We have exploited the internal structure of the coefficient matrix and derived the multistage-multigroup method.

### 3 The multistage-multigroup technique

In the multistage-multigroup least squares method the system of constraint equations is divided into stages and into groups. The matrix \( \mathbf{A}_i \) and the vector \( \mathbf{v}_i \) can be expressed as

\[ \mathbf{A}_i = [\mathbf{A}_{i,1}, \mathbf{A}_{i,2}, \ldots, \mathbf{A}_{i,j}, \ldots, \mathbf{A}_{i,p}], \quad \mathbf{v}_i = [\mathbf{v}_{i,1}^T, \mathbf{v}_{i,2}^T, \ldots, \mathbf{v}_{i,j}^T, \ldots, \mathbf{v}_{i,p}^T]^T. \]  

(7)

Assuming time step as one stage, at every stage of computation the coefficient matrix consists of three non–zero elements blocks only. The submatrices contain the internal node temperature coefficients, the boundary condition coefficients, and the thermal capacity of the control volumes, respectively. Designating these matrices by \( \mathbf{A}_{i,p}, \mathbf{A}_{i,p-1} \) and \( \mathbf{A}_{i,p-2} \), the final solution for the vector of corrections and the covariance matrices \( a \) posteriori of measurement results \( \Delta_{i,j} \) at any arbitrary \( i \)–stage can be written as:
Advanced Computational Methods in Heat Transfer

\[ \mathbf{v}^{(i)}_j = F_j \mathbf{L}_i,j \left( \mathbf{w}_i - A_{i,g} \mathbf{v}_{i-1,g} \right) \quad \text{for } j = 1, 2, ..., g+2 \]  

(8)

for the temperature field (groups \( g, g+2 \))

\[ \Delta_{i,g} = \Delta_{i-1,g} - F_g \left[ \mathbf{L}_{i,g} A_{i,g} - \mathbf{L}_{i,g} \mathbf{P}_i \mathbf{L}^T_{i,g} \right] F_g \]  

(9)

\[ \Delta_{i,g+2} = F_{g+2} - F_{g+2} \mathbf{L}_{i,g+2} A_{i,g+2} \mathbf{F}_{g+2} \]  

(10)

\[ \mathbf{L}_{i,g} = -\mathbf{L}_{i-1,g} \mathbf{P}_i \mathbf{D}_i^{-1} + A_{i,g} \mathbf{D}_i^{-1} \]  

(11)

for the boundary conditions (group \( g \))

\[ \Delta_{i,g+1} = F_{g+1} \]  

(12)

\[ \mathbf{L}_{i,g+1} = A_{i,g+1} \mathbf{D}_i^{-1} \]  

(13)

\((i = 1, 2, 3, ..., m \text{ gives } g = 1, 3, 5, ..., p-2, \text{ respectively})\). The vector of corrections and the covariance matrices \( a \text{ posteriori} \) of measurement results \( \Delta_{i,j} \) after any arbitrary \( k \)-stage is given by:

\[ \mathbf{v}_{k,i} = \sum_{i=1}^{k} \mathbf{v}^{(i)}_j \quad \text{for } i = 1, 2, ..., k \]  

(14)

\[ \Delta_{i,j} = \Delta_{i-1,j} + F_j \mathbf{L}_{i,j} \mathbf{P}_i \mathbf{L}^T_{i,j} \mathbf{F}_j \quad \text{for } j = 1, ..., g-1 \]  

(15)

\[ \mathbf{L}_{i,j} = -\mathbf{L}_{i-1,j} \mathbf{P}_i \mathbf{D}_i^{-1} \quad \text{for } j = 1, ..., g-1 \]  

(16)

The auxiliary matrices are expressed by

\[ \mathbf{D}_i = \sum_{i=g}^{g+2} A_{i,j} \mathbf{F}_j A_{i,j}^T - \mathbf{P}_i \mathbf{D}_i^{-1} \mathbf{P}_i^T \]  

\[ \mathbf{P}_j = A_{i,g} \mathbf{F}_g A_{i-1,g}^T \]  

(17)

In equations (8) to (17) \( F_j \) denotes the covariance matrix \( a \text{ priori} \). The initial values (for \( i = 0 \)) of matrices in equations (17) are equal to zero. In the MM procedure the solution is yielded hierarchically, from stage to stage. The system of constraint equations is solved using the smaller
dimension matrices. The division in groups and stages means that in the multistage-multigroup method there is no need to estimate the values and uncertainties of all unknowns at the start of the computation. These values can be predicted at every time step based on the mathematical model of the considered process.

4 Examples

4.1. The overdetermined problem

We employed the MM method for determination of the unsteady temperature field in a boiler shell, which was subjected to a thermal shock. The cylinder of inside radius of 0.1085 m and outside radius of 0.1385 m was sufficiently long that the temperature field could be considered as one-dimensional. We assumed that thermophysical properties of the shell remained constant. The shell had a thermal conductivity of 44 W/m·K and a thermal diffusivity of $1.2 \cdot 10^{-5}$ m²/s. The model of the temperature field within the shell in cylindrical coordinates consists of the governing equation

$$\frac{\partial T}{\partial \tau} = \frac{\alpha}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right)$$  \hspace{1cm} (18)

and the initial-boundary conditions

$$T(r = r_0, \tau) = T_0(\tau), \quad T(r = r_e, \tau) = T_e(\tau), \quad T(r, \tau = 0) = T_0(r)$$  \hspace{1cm} (19)

where $\alpha$, $\tau$ and $r$ denote the thermal diffusivity, time and radius of the shell, respectively. The temperatures of the internal, external surfaces and in the point situated within the area ($r = 0.1115$ m) were measured every 5 s on an experimental stand [5]. The data are presented in Figure 1. The uncertainties of the measured temperatures at the external surface of the shell were assumed as high as ± 1.0 K. The measurement of the temperature at the internal surface of the boiler shell (exposed to the hot water) and in the point within was very difficult to perform. Due to the technical difficulties in measuring the temperature in those points, the data obtained there were subjected to a greater uncertainty. We assumed it as high as ± 3.0 K. Because the boundary conditions at the surfaces of the
shell and the temperature in the point within the area were known the problem considered was overdetermined.

![Figure 1. The measured temperatures](image1)

The governing equation of the model was approximated with the Finite Volume Method (FVM). On the base of the FVM the constraint equations and the elements of coefficient matrices were derived. The uniform initial temperature within the cylinder was assumed as high as 16.0 ± 1.0 °C.

![Figure 2. The estimated temperature distribution within the boiler shell](image2)
The estimation of the temperature field was performed using the MM method. The preliminary values of the unknown temperatures at the successive stages of calculation were evaluated on the base of their physically feasible range. This was easy because the boundary conditions at the internal and external surfaces were known. Figure 2 presents the estimated temperature distribution within the boiler shell. The uncertainties a posteriori of all estimated temperatures are less than ± 0.9 K.

4.2. The ill-posed problem

We examined the efficiency and stability of the MM approach considering the 2D unsteady temperature field in another boiler, which external surface was thermally insulated. The data were known in just few points at the internal and external surfaces of the shell (Figure 3). The problem considered was ill-posed. The measurement was made on an experimental stand [5]. The temperatures in 10 points of the shell were measured every 6 s. We assumed that the cylinder was sufficiently long to consider the temperature field as 2D only (we assumed that there were no heat flow in the direction parallel to the boiler axis) and the thermophysical properties remained constant. The shell had a thermal conductivity of 44 W/m·K and a thermal diffusivity of $1.2 \cdot 10^{-6}$ m$^2$/s.

![Figure 3. The boiler shell [5]](image)

The model of the unsteady temperature field contains the governing equation (in the cylindrical coordinates system)
Advanced Computational Methods in Heat Transfer

\[
\frac{\partial T}{\partial \tau} = \alpha \cdot \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \varphi^2} \right),
\]

(20)

and the initial–boundary conditions:

\[
T(r = r_e, \varphi, \tau) = T_e(\varphi, \tau), \quad T(r = r_e, \varphi, \tau = 0) = T_0(\varphi, r) \\
T(r = r_i, \varphi, \tau) = T_i(\varphi, \tau).
\]

(21)

Figure 4. The measured temperatures

The constraint equations were derived using FVM. We assumed the uniform initial temperature within the area as high as 20 ± 1.0 °C and uncertainties of the data at the external and internal surfaces as high as ± 1.0 K and ± 3.0 K, respectively. At the other time steps (during the computation) the unknown temperatures were preliminary evaluated using the model equations. This method of evaluation allowed one to achieve the stable, correct from the physical point of view and more accurate solution, despite the problem considered being ill-posed. The uncertainties a posteriori of all variables are less than ± 2.8 K. The heat flow lasted for about 6600 s and after that time approached the quasi-steady level. Figures 5 and 6 present the estimates for the temperature distribution in selected points of the boiler.
5 Conclusions

We have demonstrated that the multistage-multigroup least squares method employed for modeling of unsteady temperature fields using data subject to uncertainties allows us to solve the adjustment problems in an
effective way. The use of MM method makes approach more consistent numerically, reduces the calculation time, and allows on-line estimation of temperature fields in the elements of technological systems. In the MM technique there is no need to preliminary estimate the values of all unknowns at the start of computation. It can be performed during the calculations based on the model equations or their physically feasible range. This gives a significant improvement to the stability and accuracy of the results and allows us to solve both well- and ill-posed heat conduction problems.

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References


