Dynamic thermoelasticity problem for a plate with a moving semi-infinite cut

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Abstract

The heat-shock, i.e. action of stresses created by a sharp change in temperature of a thin plate with a semi-infinite cut, whose tip is moving at a constant velocity since the initial time is considered. The lateral surfaces of the plate are subjected to a linear heat transfer by radiation to the surroundings.

1 Introduction

In a thin isotropic infinite plate of \(2\delta\) thickness along the ray of \(y' = 0, \ x < 0\) there is a semi-infinite cut that starts at the initial time to move to the region of \(y' = 0, \ x' > 0\) at a constant velocity. The plate heats symmetrically with respect to the middle plane by heat exchange with the medium for temperature \(T_c\) which surrounds its lateral surfaces. The initial temperature \(T_0\) of the plate is not equal to the surrounding temperature. In no time the temperature \(T_1\) arises on the edges of the cut.

It is considered that the origin of the fixed \((x', y')\)-system is at the point where the end of the cut is at the initial time. The origin of the moving \(x, y\)-coordinate system is chosen to remain at the cut tip, i.e.,

\[
x = x' - V_t t, \quad y = y', \quad t = t'
\]

This problem represents a case of plane-stress state, a case of plane strain which is realized by zero heat exchange with the medium and by replacing of elastic constants combinations. In consequence of the symmetry with respect to the \(X\)-axis, the problem is considered for a half-plane \(y \geq 0\) with the boundary and initial conditions given in the moving system of coordinates.

2 Mathematical statement of the problem.
Mathematical problem has the form,
\[ T(x, y, t) = T_0, \quad x < 0, y = 0 \] (1)
\[ \frac{\partial T(x, y, t)}{\partial y} = 0, \quad x > 0, y = 0 \] (2)
\[ T(x, y, t) = T_0, \quad t = 0 \] (3)
\[ \sigma_{yy}(x, y, t) = 0, \quad x < 0, y = 0, t > 0 \] (4)
\[ \sigma_{xy}(x, y, t) = 0, \quad -\infty < x < \infty, y = 0, t > 0 \] (5)
\[ U_y(x, y, t) = 0, \quad x > 0, y = 0, t > 0 \] (6)
\[ U_y(x, y, t) = 0, \quad t \leq 0 \] (7)
\[ \frac{\partial U_y(x, y, t)}{\partial t} = 0, \quad t \leq 0 \] (8)

3 Solution of the thermoelasticity problem

The problem formulated in eqns (1)-(3) for \( T_0 = T_c \) was solved by Zhornik & Kartashov [1] considering an analogous thermoelastic problem in the quasi-static statement. For the same reason the solution for temperature field will be the same and has the form,
\[ T(x, y, t) - T_c = \theta(x, y, t) \cdot e^{-\gamma x} + (T_0 - T_c) \cdot e^{\frac{\alpha x^2}{2t}} \] (9)
\[ \theta(x, y, t) \] is a function which has the following form according to the Laplace-Fourier transformations,
\[ \theta(\xi, y, s) = \int_{-\infty}^{\infty} e^{-ixy} dx \int_{0}^{\infty} e^{-st} \theta(x, y, t) dt = \sqrt{2\pi} \]
\[ = \left( \frac{T_1 - T_c}{s} - \frac{T_0 - T_c}{s + k \cdot \chi^2} \right) \cdot e^{-\frac{i\pi}{4}} \cdot \sqrt{i\gamma + \beta} \cdot e^{-\sqrt{\chi^2 + \rho^2} |y|} \sqrt{2\pi} \times \]
\[ \times (-\xi + i\gamma) \cdot \sqrt{-\xi + i\beta} \]
where \( \gamma = V_T / 2k, \) \( k = \lambda / \rho c \) - thermal diffusivity of the plate; \( \lambda \) - its heat conduction; \( \rho \) - density; \( c \) - specific heat;
\[ \beta = \sqrt{\chi^2 + s / (k)}, \quad \chi^2 = \gamma^2 + \chi^2, \quad \chi^2 = \alpha / \lambda \delta \]
\( \alpha \) - the coefficient of linear heat transfer to the medium embracing the lateral surfaces of the plate. The case of the stationary problem of heat conduction for \( t \to \infty \) \( (s \to 0) \) was investigated by Salganik & Chertkov [2] and the problem of heat conduction for a fixed cut was considered by Poberezhny & Gaivas [3], Kozlov, Mazya & Parton [4].
The solution of the dynamic problem of thermoelasticity will have the form
\[ \sigma_{ij} = \sigma_{ij}^T + \sigma_{ij}^P \]  
\[ U_i = U_i^T + U_i^P \]

\( \sigma_{ij}^T \) and \( U_i^T \) - the problem of thermoelasticity for a plate without cut satisfies the boundary conditions of eqns (5), (7), (8) and its solution is discovered by means of the thermoelastic potential of displacement \( F(x, y, t) \) in the \( x, y \)-coordinates from the equation below

\[ \left(1 - \frac{\alpha^2}{d^2}\right) \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - (1 + \nu) \alpha_T \theta(x, y, t) e^{-\kappa t} = 0 \]  
\[ = a^2 \left[ -\frac{2}{d} \frac{\partial^2 F}{\partial x \partial t} + \frac{\partial^2 F}{\partial t^2} \right] \]

where \( \alpha = 1/c_\| = \sqrt{\rho (1 - \nu)/2G} \) - longitudinal wave slowness; \( d = 1/V_T \) - slowness of the cut end; \( G \) - the shear modulus; \( \nu \) - Poisson’s ratio; \( \alpha_T \) - the coefficient of linear thermal expansion.

The normal stress which is necessary for the subsequent solution formulated according to Laplace-Fourier has the form

\[ \sigma_T^T(\xi, y, s) = \sqrt{\frac{2}{\pi}} G(1 + \nu) \alpha_T \left( \frac{T_1 - T_c}{s} - \frac{T_0 - T_c}{s + k\chi^2} \right) \times \]

\[ \frac{\sqrt{-i\sqrt{\gamma + \beta} \left[ \xi^2 - b^2(-\xi - i d s)^2/2d^2 \right]} \times \left[ e^{-\gamma_1|y|} - \frac{\gamma_1 e^{-a|y|}}{\sqrt{1 - \frac{a^2}{d^2}(-\xi + i s a_2)^{1/2}(-\xi - i s a_1)^{1/2}}} \right] \]

where \( \eta \to 0 \), \( \gamma_1 = \sqrt{(\xi + i \gamma)^2 + \beta^2} \), \( \alpha = \sqrt{1 - a/d} \sqrt{1 + a/d(-\xi + i s a_2)^{1/2}(-\xi - i s a_1)^{1/2}} \), \( a_1 = a/(1 + a/d), a_2 = a/(1 - a/d) \).
The boundary conditions for $\sigma_{ij}$, $U_i^p$ - solutions of isothermic theory of elasticity have the form of eqns (5)-(8), and also
\[
\sigma_{yy}^p(x, y, t) = -\sigma_{yy}^T(x, y, t), \quad x < 0, \ y = 0
\]
(15)

To find solutions for $\sigma_{ij}^p$ and $U_i^p$ it is considered a fundamental solution $\sigma_{yy}^*, U_i^*$ of elasticity theory about a stressed state of the plate with a semi-infinite cut moving at a constant velocity $V_T$ when normal concentrated forces $f$, moving subsequently at a constant distance $l$ from the cut end, instantly apply on the edges of the cut at the distance $l$ from its end. The problem as in the preceding item is considered in the moving $x,y$-coordinates of eqns (5)-(8) and also
\[
\sigma_{yy}^*(x, y, t) = -f\delta(x + l)H(t), \quad x < 0, \ y = 0
\]
(16)

where $\delta(x)$ - delta Dirac function; $H(t)$ - Heaviside function. For solution of this problem it is used an original method proposed by Freund [5] for a fixed cut.

The stress intensity factor $K_i(t)$ for $\sigma_{yy}^*(x, 0, t)$ which is necessary for further consideration, has the form
\[
K_i(t) = \sqrt{2/\pi l} \frac{f}{\pi} \left(1 - a/d\right)^{1/2} \frac{1}{\pi} \int_{a_1}^{a_2} \frac{(h - a_2)^{1/2}}{(l - h)^{1/2}} \times
\]
\[
\times (c_2 - h) S_+(-h) \right] dh \ H(t - a_2 l)
\]
(17)

where $S_+(\lambda) = \exp\left\{-\frac{1}{\pi} \int_{a_1}^{b_1} \arctan\left[4 \eta^2 |\alpha(\mp \eta)| \times
\right.\right.
\]
\[
\times \left. |\beta(\mp \eta)| \sqrt{2 \eta^2 - b^2 - \frac{b^2}{d^2} \eta^2 \mp 2 b^2 \eta \frac{d}{d}} \right] \frac{d\lambda}{\eta \pm \lambda}\}
\]
(18)

$\alpha(\lambda) = \left(1 + a/d\right)^{1/2} \left(1 - a/d\right)^{1/2} (a_1 - \lambda)^{1/2} (a_2 + \lambda)^{1/2}$

$\beta(\lambda) = \left(1 + b/d\right)^{1/2} \left(1 - b/d\right)^{1/2} (b_1 - \lambda)^{1/2} (b_2 + \lambda)^{1/2}$

$b_{1,2} = b/(1 \pm b/d)$

$c_2 = c / (1 - c / d), \quad c = 1 / c_R, \quad c_R$ - Rayleigh wave velocity.
The dependence $S_+\left(-\frac{1}{V}\right)$ on $Vb$ for various cut tip velocities $b/d$ was obtained by Zhornik & Kartashov [6]. Here is $V$, velocity of edge dislocation which is moving in the positive direction of the $x$-axis and starts the motion from the cut tip. The dislocation velocity is measured within $0 \leq V < c_{Hf} - V_T$. The curves for a fixed cut $b/d = 0$ are given by Parton & Boriskovsky [7].

In case when the load $-\sigma_{yy}(x,0,t)$ is applied to the cut edges, the stress intensity factor is represented as

$$K_i^d(t) = \int_{-\infty}^{0} dx \int_{0}^{t} \frac{\partial \sigma_{yy}(x,0,\tau)}{\partial \tau} K_i(x,t-\tau) d\tau$$

(19)

where $K_i(x,t)$ results from eqn (17) by replacing $l$ with $-x > 0$, and we also take $f = 1$. Then to eqn (19) the Laplace transform is applied

$$\tilde{K}_i^d(s) = \int_{-\infty}^{0} dx \sigma_{yy}(x,0,s) K_i(x,s)$$

(20)

And then using the Parceval ratio by Sneddon [8] $\tilde{K}_i^d(s)$ has the form

$$\tilde{K}_i^d(s) = \int_{-\infty}^{\infty} s \sigma_{yy}(\xi,0,s) K_i(-\xi,s) d\xi$$

(21)

where $\sigma_{yy}(\xi,0,s)$ is given in eqn (14) for $T_0 = T_c$, $K_i(-\xi,s)$ is obtained from eqn (17) using the above-mentioned replacements and applying to eqn (17) the Laplace-Fourier transform with parameters of $s$ and $-\xi$ and it will have the form

$$K_i(-\xi,s) = -\frac{i}{\sqrt{\pi}} \left(1 - \frac{a}{d}\right) \frac{(i\xi - a_2 s)^{1/2}}{s(c_2 s + i\xi) S_+(i\xi/s)}$$

(22)

The substitution of eqn (14) and (22) into eqn (21) comes $\tilde{K}_i^d(s)$ to the form

$$\tilde{K}_i^d(s) = -2\pi Aa^{3/2}b^2 \sqrt{1 + a/d} / (\sqrt{\beta^2 + \gamma^2 + as}) \sqrt{s S_+(0) + \frac{Aa^2 \sqrt{\gamma + \beta}}{s} \int_{-\infty}^{\infty} \left\{ \xi^2 + \eta_i^2 \right\} / \sqrt{\gamma + \beta + i\xi} \times \left[ \sqrt{(-\xi + i\gamma)^2 + \beta^2 + \eta_i} \right] \sqrt{a_i s - i\xi \left(s + \frac{i\xi}{c_2} \right) S_+ \left(\frac{i\xi}{s} \right)} \right\} d\xi,$$
For a fixed cut \( d \to \infty \) eqn (23) is obtained by general asymptotic method by Kozlov, Mazya & Parton [9]. In expression of eqn (23) integration is conducted along the real axis embracing the origin of the coordinates from below.

Calculation of \( K^d_i(s) \) is connected with calculating of the contour integral in the lower half-plane of \( \xi \), which embraces the cut along the imaginary axis from \(-iaS\) till \(-i(\beta - \gamma)\).

Having made the integration and proceeding to the original of \( t \), \( K^d_i(t) \) for large and little intervals comes to the form

\[
K^*_i = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} (2\gamma^*)^{-k-1} \frac{\Gamma\left( k - \frac{1}{2} \right)}{\Gamma(k+1)} \left( \frac{1}{2} - \frac{1}{2} \right)^m \frac{(-1)^m}{m!} (\gamma^*)^{2m} \times
\]

\[
\frac{1}{2} + m \left( k \right)^{1/2} \times
\]

\[
\gamma^* \times
\]

\[
\tau >> a^2 k^2 \left( 1 + \sqrt{1 + 4 \chi^2 a^2 k} \right) / 2\delta^2
\]
Figure 1: Relation between $K_i^*$ and $\tau$ for various $\gamma^*$

- $\chi^* = 0$
- $\chi^* = 0.5$

Figure 2: Relation between $K_i^*$ and $\tau$ for various $\gamma^*$

- $\chi^* = 0.1$
- $\chi^* = 1$
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\[ \tau >> a^2 k^2 \left( 1 + \sqrt{1 + 4 \chi^2 a^2 k} \right) / 2 \delta^2 \]

\[ K_i^* = -K_i^d / \sqrt{2} G (1 + \nu) \alpha_\gamma (T_i - T_c) \sqrt{\delta / 1 - \alpha / d} \]

\[ \tau = k t / \delta^2 \text{-the Fourier criterion; } \gamma^* = \gamma \delta, \chi^* = \chi \delta, F_2(a, b; c, d; e) \]

is the generalized hypergeometrical function; \( \Gamma(a) \) is the gamma function.

The dependence of \( K_i^* \) on \( \tau \) for great intervals under different intensities of heat exchange \( \chi^* \) and moving cut velocities \( \gamma^* \) is given on Fig. 1 - 2.

For \( \gamma^* = 0 \) case eqns (24),(25) were obtained by Kozlov, Mazya & Parton [9], eqn (25) for \( d = 0 \) was obtained by Zhornik & Kartashov [1].

References


