Solving incompressible fluid flow problem by the dual reciprocity method with new approximation functions

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Abstract

In the paper the Dual Reciprocity BEM with the modified approximation functions is used to obtain velocity field which can be then utilized when solving convection-diffusion equation. Velocity-pressure formulation (primitive variables) is considered. Solution of the Navier-Stokes equation for incompressible fluid is achieved iteratively. For a given pressure field, velocity distribution is calculated within the domain. In next step pressure field is updated in accordance to the differential equation derived from the continuity condition.

1 Introduction

Heat convection is an example of coupled boundary value problem. Determining temperature field one has to solve convection-diffusion equation in which convective term plays an important role and may not be neglected. This term expresses the influence of the velocity field on fluid temperature. Thus, the knowledge of velocity profiles within the fluid is essential as the fluid flow problem is primary to convective heat transfer.

Velocity field is a solution of Navier-Stokes equation. In order to eliminate pressure gradients, stream function and vorticity can be introduced into boundary problem. As a result the velocity-vorticity formulation is obtained. Such a formulation was already widely used in FDM, FEM and BEM contexts, e.g. [1, 2, 4]. The velocity-pressure formulation (primitive variables) requires the momentum equation to be supplemented by the continuity equation. Pressure gradients substituted into the Navier-Stokes equation should cause that the velocity field found from this equation, simultaneously has to satisfy the continuity condition. Solution is approached in iterative process and many versions of SIMPLE algorithm, which is used to correct pressure field, have already been published, e.g. [1, 2, 5]. In BEM context the velocity-pressure formulation was recently used by Darkovich, Kakuda and Tosaka [6].

In this paper, as well as in the paper published in 1992 [12] such formulation was employed to solve incompressible 2-D fluid flow problem. However, in order to supplement Navier-Stokes equation the Poisson differential equation [7] is used. The problem is also solved iteratively. Assuming both velocity and pressure fields (as an initial guess), the Navier-Stokes equation is solved and new velocity profiles are determined within the fluid. From the Poisson equation pressure distribution is corrected and both fields are updated. Once the convergence criterion is fulfilled iteration loop is completed.

Solving Navier-Stokes and Poisson differential equations the Boundary Element Method is applied [8, 9]. Both equations are transformed into integral equations and then solved
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numerically. Domain integrals occurring in these equations are converted to the boundary ones employing the Dual Reciprocity Method [10].

In the previous paper [12], we have assumed classical DRM interpolation functions depending on the distance between collocation and field points only. After carrying out several test calculations we have noticed poor convergence or even divergence in the cases of large Reynolds number. Recently new functions were worked out by Goldberg and Chen [3], which are the most appropriate for 2D global approximation. We have also changed the way of pressure Poisson equation handling. In our opinion it should increase the convergence in the case of larger Reynolds numbers.

2 Formulation of the Problem

Incompressible fluid flow problem is governed by the Navier-Stokes differential equation which in vector notation can be written as

\[ \nabla^2 \vec{u} = \frac{\rho}{\eta} \cdot \nabla \vec{u} + \frac{1}{\eta} \nabla p - \frac{\rho}{\eta} \vec{g} \quad (1) \]

where \( \vec{u} \) means the fluid velocity, \( \rho \) stands for the density, \( \eta \) is the fluid viscosity and the pressure is represented by \( p \). Vector \( \vec{g} \) designates the gravitational acceleration.

Together with the Navier-Stokes equation one has to consider the continuity condition which has the following form

\[ \nabla \cdot \vec{u} = 0 \quad (2) \]

Differentiating the momentum equation (1) and making use of the continuity condition (2) one arrives at the following Poisson equation for pressure field [7]

\[ \nabla^2 p = -2 \rho \left( \frac{\partial u_x}{\partial y} \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial x} \frac{\partial u_y}{\partial y} \right) \quad (3) \]

where \( u_x \) and \( u_y \) are components of vector \( \vec{u} \).

In order to obtain unique solution of the above boundary problem the appropriate boundary conditions have to be prescribed. They are in usual form, i.e. velocity \( \vec{u} \) is specified along boundaries of the region and pressure distribution is known in at least one point.

Upon defining the fundamental solution \( u^* \) as a potential generated by Dirac's delta function \( \delta_i \) acting at point \( i \)

\[ \nabla^2 u^* = \delta_i \quad (4) \]

and applying the reciprocity theorem [8, 9] one can transform the Navier-Stokes equation into following integral equation

\[ c_i \vec{u}_i + \int_{\Gamma} q^* \vec{u} d\Gamma = \int_{\Gamma} u^* \vec{q}_u d\Gamma + \frac{\rho}{\eta} \int_{\Omega} u^* \vec{u} \cdot \nabla \vec{u} d\Omega + \]

\[ + \frac{1}{\eta} \int_{\Omega} u^* \nabla \vec{p} d\Omega - \frac{\rho}{\eta} \int_{\Omega} u^* \vec{g} d\Omega \quad (5) \]
Quantities $q^*$ and $\tilde{q}_u$ are flux analogs as they are derivatives of $u^*$ and $\tilde{u}$, respectively along outward normal, i.e.

$$q^* = -\frac{\partial u^*}{\partial n}$$

$$\tilde{q}_u = -\frac{\partial \tilde{u}}{\partial n}$$

Symbol $\frac{\partial}{\partial n}$ stands for differentiation along outward normal.

The same procedure applied to Poisson equation (3) yields

$$c_i p_i + \int_\Gamma q^* p d\Gamma = \int_\Gamma u^* q_p d\Gamma - 2 \int_\Omega u^* \left( \frac{\partial u_x}{\partial y} \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial x} \frac{\partial u_y}{\partial y} \right) d\Omega$$

where flux analog $q_p$ is defined as

$$q_p = -\frac{\partial p}{\partial n}$$

Integral equations (5) and (8) constitute the system of equations fully equivalent to the primary boundary problem (1) and (2). They can be discretized and solved numerically. However, they contain both boundary as well as domain integrals and in order to avoid domain discretization, domain integrals will be converted to the boundary using the Dual Reciprocity Method [10].

3 Transformation of the Domain Integrals to the Boundary

The Dual Reciprocity Method is a general method of transforming domain integrals, having the form

$$D = \int_\Omega b u^* d\Omega$$

to the equivalent boundary ones. The method is based on the approximation of function $b$ using the global approximation functions $f^j$ [10]

$$b = \sum_{j=1}^{NP} f^j \alpha^j = \sum_{j=1}^{NP} \nabla^2 \tilde{u}^j \alpha^j$$

The total number of terms in summation (11) $NP = N + P$, where $N$ is a number of boundary nodes along surface $\Gamma$ and $P$ is a number of internal poles selected inside the domain $\Omega$ in order to increase the accuracy of the interpolation. Function $\tilde{u}^j$ is a particular solution of the following Poisson equation

$$\nabla^2 \tilde{u}^j = f^j$$

Constant coefficients $\alpha^j$ are determined upon condition that the relationship (11) holds at all boundary nodes and internal poles, i.e.

$$\alpha = F^{-1}B$$

where $F$ is a square matrix formed of interpolation functions and vector $B$ contains the values of function $b$ at boundary nodes as well as internal poles. Previous works on dual
reciprocity have shown that although a variety of functions can in principle be used as a basic approximation function, good results were usually obtained with simple expansions, the most popular of which is \( f = 1 + R \), where \( R \) is the distance between prespecified fixed collocation points, \( y \), and a field point \( x \) where the function is approximated. In the DRM literature given, the choice is based on experience rather than formal mathematical analysis.

However, as it is suggested in [3] the best possible approximating functions for \( b \) in equation (11) is given by

\[
b(x) = \sum_{j=1}^{NP} \alpha_j R^2 \log R + a x + b y + c
\]

The set of equations (14) allows one to determine the components of vector \( B \) for each boundary node and each internal point.

Introducing the boundary element influence matrices the domain integral (10) can be expressed as

\[
D = (H \tilde{U} - G \tilde{Q}) \alpha
\]

where matrices \( \tilde{U} \) and \( \tilde{Q} \) contain functions \( \tilde{u}^j \) and its normal derivative \( \tilde{q}^j = -\frac{\partial \tilde{u}^j}{\partial n} \), respectively.

Following this idea, the subsequent terms in integral equations (5) and (8), can be written as

- convective term in Navier-Stokes equation

\[
\int \Omega^* \tilde{u} \cdot \nabla \tilde{u} d\Omega = (H \tilde{U} - G \tilde{Q}) \alpha_{NS}
\]

Vector containing the values of \( \alpha_{NS} \) can be derived from interpolation (14).

- pressure gradient term in Navier-Stokes equation

\[
\int \Omega^* \frac{\partial p}{\partial x} d\Omega = (H \tilde{U} - G \tilde{Q}) F^{-1} F_x \alpha_p
\]

\[
\int \Omega^* \frac{\partial p}{\partial y} d\Omega = (H \tilde{U} - G \tilde{Q}) F^{-1} F_y \alpha_p
\]

where vector \( \alpha_p \) can be derived from interpolation (14)

- gravitational acceleration term in Navier-Stokes equation

\[
\int \Omega^* \tilde{g} d\Omega = (H \tilde{U} - G \tilde{Q}) \alpha_g
\]

- source term in Poisson equation

\[
\int \Omega^* \left( \frac{\partial u_x}{\partial y} \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial x} \frac{\partial u_y}{\partial y} \right) d\Omega = (H \tilde{U} - G \tilde{Q}) \alpha_{pe}
\]

where vector \( \alpha_{pe} \) can be derived from interpolation (14)
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It should be noted that different interpolation functions can be employed for different fields. In this paper however, the same approximation was used throughout the analysis.

Introducing Eqs(16)-(20) into Eqs(5) and (8) one arrives at the following system of algebraic equations for velocity components \( u_x \) and \( u_y \) and pressure \( p \):

\[
\begin{align*}
H U_x &= G Q_{u,x} + \frac{\rho}{\eta} (H \hat{U} - G \hat{Q}) \alpha_{NS} + \\
&+ \frac{1}{\eta} (H \hat{U} - G \hat{Q}) \alpha_p - \frac{\rho}{\eta} (H \hat{U} - G \hat{Q}) \alpha_{xz} \\
H U_y &= G Q_{u,y} + \frac{\rho}{\eta} (H \hat{U} - G \hat{Q}) \alpha_{NS} + \\
&+ \frac{1}{\eta} (H \hat{U} - G \hat{Q}) \alpha_p - \frac{\rho}{\eta} (H \hat{U} - G \hat{Q}) \alpha_{yy} \\
H P &= G Q_p - 2 (H \hat{U} - G \hat{Q}) \alpha_{pe}
\end{align*}
\] (21)

4 Solution Procedure

The set of algebraic equations (21)-(23) constitutes the coupled nonlinear system which can only be solved iteratively. The unknowns in this system are velocity components \( u_x \) and \( u_y \), their normal derivatives \( q_{u,x} \) and \( q_{u,y} \), pressure \( p \) and its normal derivative \( p_n \). Functions \( u_x, u_y \) and \( p \) have to be determined at all boundary nodes and internal poles whereas normal derivatives \( q_{u,x}, q_{u,y} \) and \( q_p \) exist only on the boundary.

In order to start the iterative loop both velocity as well as pressure field have to be assumed. The better this guess is, the less iterations are required to reach the solution. Knowing velocity and pressure fields all terms in Eqs(21)-(23), resulting from domain integration, can be calculated. Upon introducing boundary conditions, Eqs(21) and (22) are rearranged into the system of linear equations from which velocity components at internal poles and derivatives of velocity on the boundary are obtained. This allows to update the velocity field.

Eq.(23) is used to correct pressure distribution. The first step is to express the normal derivative of pressure \( q_p \) in terms of approximation functions \( f^j \) and coefficients \( \alpha^j \). This can be done employing the approximation formula (11). After differentiation and simple algebra manipulation one arrives at

\[
Q_p = (N_x F_x + N_y F_y) \alpha
\] (24)

where diagonal square matrices \( N_x \) and \( N_y \) contain on the principal diagonals components of the outward normal vector (\( N \) non zero values) and \( P \) zeros.

Substituting Eq.(24) into (23) and introducing boundary conditions one can rearrange this equation and solve it for unknown pressure field. Having this done the pressure field can be updated using again the DRM approximation

\[
P = F \alpha
\] (25)

Iterative process can now be repeated unless convergence is reached.

Described solution procedure have been implemented and some numerical test cases are now subjects of extensive analysis. Results of performed calculations will be presented during the conference.
Acknowledgements

The financial assistance of the National Committee for Fundamental Research, Poland, within the grant no 3 P404 025 07 coordinated by European Union within the action COST512 is gratefully acknowledged herewith.

References


