Numerical identification of a source location in steady state heat conduction

K. Iijima[1], Y. Ohura[2] & K. Onishi[3]
[1] *Graduate School of Science and Engineering, c/o Professor Onishi, Ibaraki University, Bunkyo 2-1-1, Mito-shi 310-8512, Japan*
[2] *Kyushu Institute of Information Sciences, Saifu 6-3-1, Dazaifu-shi 818-0117, Japan*
[3] *Department of Mathematical Sciences, Ibaraki University, Bunkyo 2-1-1, Mito-shi, Japan*

Abstract

An identification problem is considered for detecting the location of a single heat source in a body in the steady state of heat conduction. The detection is based on the partial observation of temperatures and heat fluxes on part of the surface. The boundary element method and high order finite difference method are used for a computational tool with the benefit of their high accuracy. Numerical results are presented to show the usefulness of the proposed method for identification.

1 Introduction

Let Ω be a domain with its boundary Γ in $\boldsymbol{R}^2$ with the x_1x_2-rectangular coordinates. We denote a point in Ω by $\boldsymbol{x} = (x_1, x_2)$. The domain stands for a cross section of a heat conducting medium, and the boundary is assumed to be piecewise smooth. We denote by $\Omega_{(\alpha,\beta)}$ the domain which is obtained from Ω with an internal point (α, β) removed. We consider a harmonic function $u(\boldsymbol{x})$ on $\Omega_{(\alpha,\beta)}$, satisfying the steady state heat conduction equation

$$
\begin{aligned}
&-\Delta u(\boldsymbol{x}) = 0\,, \qquad \boldsymbol{x} \in \Omega_{(\alpha,\beta)} \\
&u = O\left(\frac{1}{r}\right) \quad \text{as} \quad r = \sqrt{x_1^2 + x_2^2} \to +\infty
\end{aligned}
\tag{1}
$$



Heat Transfer VIII, B. Sundén, C. A. Brebbia & A. Mendes (Editors)
© 2004 WIT Press, www.witpress.com, ISBN 1-85312-705-1

with $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$. We say that (α, β) is an isolated essential singular point of the harmonic function $u(\boldsymbol{x})$ on $\Omega_{(\alpha,\beta)}$ if $u(\boldsymbol{x})$ cannot be extended to a harmonic function over the whole domain Ω.

For example, the point $(\alpha, \beta) = (0, 1)$ is an isolated essential singular point of the harmonic function [1]

$$u(x_1, x_2) = \frac{1 + x_1^2 - x_2^2}{(1 + x_1^2 - x_2^2)^2 + 4x_1^2 x_2^2} \tag{2}$$

for the strip $\Omega = (-2, 2) \times (0, +\infty)$. In fact, this function Eq.(2) is the unique solution of the Cauchy problem of the Laplace equation Eq.(1) in the half space $(-\infty, +\infty) \times (0, +\infty)$ subject to the Cauchy data $u(x_1, 0) = \varphi(x_1) = \dfrac{1}{1 + x_1^2}$ and $\dfrac{\partial u}{\partial x_2}(x_1, 0) = \psi(x_1) = 0$ for $-\infty < x_1 < +\infty$. The solution is obtained immediately from the Stokes formula

$$u(x_1, x_2) = \frac{1}{2}\left\{\varphi(x_2 - ix_1) + \varphi(x_2 + ix_1)\right\} + \frac{1}{2i}\int_{x_2 - ix_1}^{x_2 + ix_1} \psi(\xi) d\xi$$

with the imaginary unit i $(i^2 = -1)$.

It is known mathematically in general that $u(\boldsymbol{x})$ must diverge to infinity with a rate of order of at least $\ln\sqrt{(x_1 - \alpha)^2 + (x_2 - \beta)^2}$ as (x_1, x_2) tends to the isolated essential singular point (α, β). Physically the isolated essential singular point implies a point heat source of some kind.

The purpose of this study is to present an idea about a possible numerical identification method for the location of the isolated essential singular point (α, β) of the harmonic function Eq.(2). We assume that the Dirichlet data and the Neumann data are available only along a part of the boundary Γ. In this paper we consider an example exclusively for the function Eq.(2). The identification method is formulated as the series of Cauchy problems of the Laplace equation. These Cauchy problems are numerically solved by two methods; the direct method of solution to the primary and the adjoint problems by using the boundary element method, and recently developed high order finite difference method [2].

2 Method of detection

Let Γ_u be a part of the boundary Γ, and let Γ_q be another part of the boundary. We reformulate the Cauchy problem of the Laplace equation Eq.(1) in somewhat general sense: The problem is to find $u(\boldsymbol{x})$ satisfying Eq.(1) subject to the conditions:

$$u(\boldsymbol{x}) = \bar{u}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_u \quad \text{and} \quad \frac{\partial u}{\partial n}(\boldsymbol{x}) = \bar{q}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma_q \tag{3}$$

for given Dirichlet data $\bar{u}$ and Neumann data $\bar{q}$. Here, $\boldsymbol{n}$ is the unit normal exterior to the boundary. The problems to be mentioned below fall into this reformulation.

Heat Transfer VIII, B. Sundén, C. A. Brebbia & A. Mendes (Editors)
© 2004 WIT Press, www.witpress.com, ISBN 1-85312-705-1

Let Ω_{sub} be a subdomain of Ω, which can be obtained from Ω by continuous deformation, as shown in Figure 1. Here we notice that $\Gamma_u \cup \Gamma_q \subset \partial\Omega_{sub}$ for the boundary of Ω_{sub}. We denote $\Gamma_{id} = \partial\Omega_{sub} \setminus (\Gamma_u \cup \Gamma_q)$. The subscript $_{id}$ to Γ refers to "to be identified".

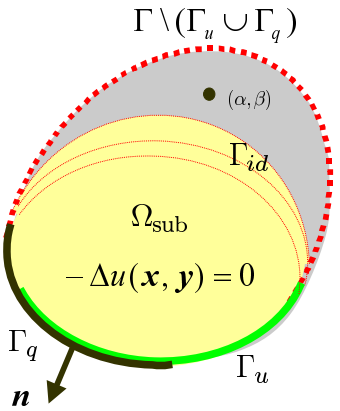


Figure 1: Continuously deformed subdomains.

If Ω_{sub} is small enough, it would not contain the isolated essential singular point (α, β). Then the Cauchy data $\bar{u}$ and $\bar{q}$ can be extended up to the boundary Γ_{id}. On the other hand, when Ω_{sub} happens to contain the singular point (α, β), the Cauchy data cannot be extended to Γ_{id}. Therefore, as the boundary Γ_{id} moves continuously, starting from sufficiently small Ω_{sub} with Γ_{id} adjacent to $\Gamma_u \cup \Gamma_q$, and just at the time when Γ_{id} crosses the singular point (α, β), a boundary value of $u(\boldsymbol{x})$ on Γ_{id} blows up to infinity at the point. If the Cauchy problem will be solved numerically, we can anticipate that approximate values of $u(\boldsymbol{x})$ on Γ_{id} at that time behaves oddly.

Our method of detection to be proposed consists of sensing some odd behaviors of the approximate values of $u(\boldsymbol{x})$ on Γ_{id} as Ω_{sub} is deformed continuously. When some odd behaviors can be caught, it only indicates that a singular point may exist somewhere on that Γ_{id}. We require another series of deformation Ω_{sub} in order to identify the location of the singular point. If we use straight lines for Γ_{id}, two family of lines must be linearly independent.

We shall consider three kinds of subdomain Ω_{sub}, and we name the corresponding Cauchy problems as follows.

Problem [A]: We denote a subdomain of Ω by $\Omega_{sub} = \Omega_a = (-2, 2) \times (0, a)$ for a real number a $(0 < a \leq 2)$. We consider a Cauchy problem of the Laplace

Heat Transfer VIII, B. Sundén, C. A. Brebbia & A. Mendes (Editors)
© 2004 WIT Press, www.witpress.com, ISBN 1-85312-705-1

equation Eq.(1) in Ω_a with the following data:

$$u(x_1,0)=\frac{1}{1+x_1^2}\,,\quad \frac{\partial u}{\partial x_2}(x_1,0)=0 \qquad (-2\le x_1\le 2)\,,$$
$$u(-2,x_2)=u(2,x_2)=\frac{5-x_2^2}{(5-x_2^2)^2+16x_2^2} \quad (0\le x_2\le a)\,.$$

Problem [B]: We take $\Omega_{sub}=\Omega_b=(-2,b)\times(0,2)$ for a real number b $(-2<b\le 2)$. We consider a Cauchy problem of the Laplace equation in Ω_b with the following data:

$$u(x_1,0)=\frac{1}{1+x_1^2}\,,\quad \frac{\partial u}{\partial x_2}(x_1,0)=0 \qquad (-2\le x_1\le b)\,,$$
$$u(-2,x_2)=\frac{5-x_2^2}{(5-x_2^2)^2+16x_2^2} \quad (0\le x_2\le 2)\,.$$

Problem [C]: We take another subdomain $\Omega_c=\Omega\cap\{\,(x_1,x_2)\mid x_2<-x_1+c\,\}$ for a real number c $(-2<c\le 4)$. We consider a Cauchy problem of the Laplace equation Eq.(1) in Ω_c with the following data:

For $-2<c\le 0$:

$$u(x_1,0)=\frac{1}{1+x_1^2}\,,\quad \frac{\partial u}{\partial x_2}(x_1,0)=0 \qquad (-2\le x_1\le c)\,,$$
$$u(-2,x_2)=\frac{5-x_2^2}{(5-x_2^2)^2+16x_2^2} \qquad (0\le x_2\le 2+c)\,.$$

For $0<c\le 2$:

$$u(x_1,0)=\frac{1}{1+x_1^2}\,,\quad \frac{\partial u}{\partial x_2}(x_1,0)=0 \qquad (-2\le x_1\le c)\,,$$
$$u(-2,x_2)=\frac{5-x_2^2}{(5-x_2^2)^2+16x_2^2} \qquad (0\le x_2\le 2)\,.$$

For $2<c\le 4$:

$$u(x_1,0)=\frac{1}{1+x_1^2}\,,\quad \frac{\partial u}{\partial x_2}(x_1,0)=0 \qquad (-2\le x_1\le 2)\,,$$
$$u(-2,x_2)=u(2,x_2)=\frac{5-x_2^2}{(5-x_2^2)^2+16x_2^2} \qquad (0\le x_2\le 2)\,.$$

3 Numerical tools

We suppose that $u(\boldsymbol{x})$ is harmonic in Ω in this section.

Heat Transfer VIII, B. Sundén, C. A. Brebbia & A. Mendes (Editors)
© 2004 WIT Press, www.witpress.com, ISBN 1-85312-705-1

3.1 Boundary element method

We denote by $\Gamma_u^c = \Gamma \setminus \Gamma_u$ and $\Gamma_q^c = \Gamma \setminus \Gamma_q$ the complement sets of Γ_u and Γ_q, respectively, with respect to the whole boundary Γ. We consider a variational problem of finding a minimizer $\omega(\boldsymbol{x})$ on Γ_u^c for the functional

$$J(\omega) = \int_{\Gamma_q} |q(\boldsymbol{x};\omega) - \bar{q}(\boldsymbol{x})|^2 d\Gamma(\boldsymbol{x}) + \eta \int_{\Gamma} |q(\boldsymbol{x};\omega)|^2 d\Gamma(\boldsymbol{x})\,, \tag{4}$$

where $q(\boldsymbol{x};\omega) = \dfrac{\partial u}{\partial n}(\boldsymbol{x};\omega)$, the normal derivative of $u(\boldsymbol{x};\omega)$, and the function $u(\boldsymbol{x};\omega)$ is a solution of the following conventional boundary value problem of the first kind for the prescribed ω:

$$-\Delta u(\boldsymbol{x};\omega) = 0\,, \quad \boldsymbol{x} \in \Omega \tag{5}$$

$$u(\boldsymbol{x};\omega) = \bar{u}\,, \quad \boldsymbol{x} \in \Gamma_u \quad \text{and} \qquad u(\boldsymbol{x};\omega) = \omega\,, \quad \boldsymbol{x} \in \Gamma_u^c\,. \tag{6}$$

The real number η in Eq.(4) works as a regularization parameter.

The solution of the problem Eqs.(5) and (6) must satisfy the boundary integral equation

$$\frac{\Theta(\boldsymbol{\xi})}{2\pi} u(\boldsymbol{\xi};\omega) + \int_{\Gamma} \frac{\partial G}{\partial n}(\boldsymbol{x},\boldsymbol{\xi}) u(\boldsymbol{x};\omega) d\Gamma(\boldsymbol{x}) = -\int_{\Gamma} G(\boldsymbol{x},\boldsymbol{\xi}) q(\boldsymbol{x};\omega) d\Gamma(\boldsymbol{x})\,, \ \boldsymbol{\xi} \in \Gamma\,. \tag{7}$$

Here, $\Theta(\boldsymbol{\xi})$ measures the internal angle (in radians) at the boundary point $\boldsymbol{\xi}$ subtending to the domain Ω, and $G(\boldsymbol{x},\boldsymbol{\xi})$ is the fundamental solution to the differential operator $-\Delta$ in two dimensions, namely $G(\boldsymbol{x},\boldsymbol{\xi}) = -\dfrac{1}{2\pi} \ln \parallel \boldsymbol{x} - \boldsymbol{\xi} \parallel$ with the distance $\parallel \boldsymbol{x} - \boldsymbol{\xi} \parallel$ between two points $\boldsymbol{x}$ and $\boldsymbol{\xi}$.

The minimizer ω of the functional Eq.(4) needs to satisfy $J'(\omega) = 0$ for the first variation $J'(\omega)\delta\omega$, which is given by the expression [3]

$$J'(\omega) = \frac{\partial v}{\partial n}(\boldsymbol{x};\omega)\,, \qquad \boldsymbol{x} \in \Gamma_u^c\,. \tag{8}$$

Here, $v(\boldsymbol{x};\omega)$ is the solution of the following boundary value problem of the first kind:

$$-\Delta v(\boldsymbol{x};\omega) = 0\,, \quad \boldsymbol{x} \in \Omega \tag{9}$$

$$v(\boldsymbol{x};\omega) = 2\left[(1+\eta) q(\boldsymbol{x};\omega) - \bar{q}(\boldsymbol{x})\right]\,, \quad \boldsymbol{x} \in \Gamma_q$$

$$\text{and} \quad v(\boldsymbol{x};\omega) = 2\eta q(\boldsymbol{x};\omega)\,, \quad \boldsymbol{x} \in \Gamma_q^c\,. \tag{10}$$

The problem Eqs.(5) and (6) is called primary problem of the variational problem with the functional Eq.(4), and the problem Eqs.(9) and (10) is called adjoint

Heat Transfer VIII, B. Sundén, C. A. Brebbia & A. Mendes (Editors)
© 2004 WIT Press, www.witpress.com, ISBN 1-85312-705-1

problem to the primary problem. The solution of the adjoint problem also needs to satisfy the boundary integral equation of the form

$$\frac{\Theta(\boldsymbol{\xi})}{2\pi}v(\boldsymbol{\xi};\omega)+\int_\Gamma \frac{\partial G}{\partial n}(\boldsymbol{x},\boldsymbol{\xi})v(\boldsymbol{x};\omega)d\Gamma(\boldsymbol{x}) = -\int_\Gamma G(\boldsymbol{x},\boldsymbol{\xi})r(\boldsymbol{x};\omega)d\Gamma(\boldsymbol{x})\,,\ \boldsymbol{\xi}\in\Gamma\,, \tag{11}$$

where $r(\boldsymbol{x};\omega)=\dfrac{\partial v}{\partial n}(\boldsymbol{x};\omega)$.

We take n nodes on the boundary Γ, and approximate the boundary by a polygon Γ^h by connecting the nodes using straight lines. Two boundary integral equations Eqs.(7) and (11) are discretized in the standard manner by using linear boundary elements into algebraic linear systems:

$$[\boldsymbol{H}]\{\boldsymbol{u}\} = -[\boldsymbol{G}]\{\boldsymbol{q}\}\,,\qquad [\boldsymbol{H}]\{\boldsymbol{v}\} = -[\boldsymbol{G}]\{\boldsymbol{r}\}\,, \tag{12}$$

respectively. Here, $[\boldsymbol{H}]$ and $[\boldsymbol{G}]$ are $n\times n$ matrices, and $\{\boldsymbol{u}\}$, $\{\boldsymbol{q}\}$, $\{\boldsymbol{v}\}$, $\{\boldsymbol{r}\}$ are n vectors with approximate nodal values of u, q, v, r, respectively.

We assume that the number of nodes belonging to Γ_u is n_1, and the number of nodes belonging to Γ_q is n_2. We set $n_1^c = n-n_1$ and $n_2^c = n-n_2$. We collect those nodal values that are on Γ_u from the n vector $\{\boldsymbol{u}\}$ to get an n_1 vector $\{\boldsymbol{u}_1\}$, leaving other nodal values on Γ_u^c to form an n_1^c vector $\{\boldsymbol{u}_2\}$. Likewise, we collect those nodal values that are on Γ_q from the n vector $\{\boldsymbol{q}\}$ to get an n_2 vector $\{\boldsymbol{q}_2\}$, leaving other nodal values on Γ_q^c to form an n_2^c vector $\{\boldsymbol{q}_1\}$. The partition can be described in the forms

$$\{\boldsymbol{u}\} = \begin{Bmatrix} \boldsymbol{u}_1 & \text{on} & \Gamma_u \\ \boldsymbol{u}_2 & \text{on} & \Gamma_u^c \end{Bmatrix} \begin{matrix} n_1 \\ n_1^c \end{matrix}\,,\qquad \{\boldsymbol{q}\} = \begin{Bmatrix} \boldsymbol{q}_1 & \text{on} & \Gamma_q^c \\ \boldsymbol{q}_2 & \text{on} & \Gamma_q \end{Bmatrix} \begin{matrix} n_2^c \\ n_2 \end{matrix}\,. \tag{13}$$

The similar partition is done for vectors $\{\boldsymbol{v}\}$ and $\{\boldsymbol{r}\}$. We have

$$\{\boldsymbol{v}\} = \begin{Bmatrix} \boldsymbol{v}_1 & \text{on} & \Gamma_q^c \\ \boldsymbol{v}_2 & \text{on} & \Gamma_q \end{Bmatrix} \begin{matrix} n_2^c \\ n_2 \end{matrix}\,,\qquad \{\boldsymbol{r}\} = \begin{Bmatrix} \boldsymbol{r}_1 & \text{on} & \Gamma_u \\ \boldsymbol{r}_2 & \text{on} & \Gamma_u^c \end{Bmatrix} \begin{matrix} n_1 \\ n_1^c \end{matrix}\,. \tag{14}$$

Corresponding to the partition described in Eqs.(13) and (14), the linear systems Eq.(12) can be written in the partitioned forms

$$\begin{matrix} & n_1 & n_1^c \\ n & \end{matrix} \begin{bmatrix} \boldsymbol{H}_1^{(1)} & \boldsymbol{H}_2^{(1)} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{Bmatrix} = -\begin{matrix} & n_2^c & n_2 \\ & \end{matrix} \begin{bmatrix} \boldsymbol{G}_1^{(1)} & \boldsymbol{G}_2^{(1)} \end{bmatrix} \begin{Bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \end{Bmatrix}, \tag{15}$$

$$\begin{matrix} & n_2^c & n_2 \\ n & \end{matrix} \begin{bmatrix} \boldsymbol{H}_1^{(2)} & \boldsymbol{H}_2^{(2)} \end{bmatrix} \begin{Bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{Bmatrix} = -\begin{matrix} & n_1 & n_1^c \\ & \end{matrix} \begin{bmatrix} \boldsymbol{G}_1^{(2)} & \boldsymbol{G}_2^{(2)} \end{bmatrix} \begin{Bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{Bmatrix}. \tag{16}$$

The boundary conditions Eqs.(6) and (10) are paraphrased in terms of approximate nodal values in the forms

$$\{\boldsymbol{u}_1\} = \{\bar{\boldsymbol{u}}_1\}\,,\quad \{\boldsymbol{u}_2\} = \{\boldsymbol{\omega}\}\,,$$

Heat Transfer VIII, B. Sundén, C. A. Brebbia & A. Mendes (Editors)
© 2004 WIT Press, www.witpress.com, ISBN 1-85312-705-1

$$\{\boldsymbol{v}_2\} = 2\left((1+\eta)\{\boldsymbol{q}_2\} - \{\bar{\boldsymbol{q}}\}\right) , \quad \{\boldsymbol{v}_1\} = 2\eta\{\boldsymbol{q}_1\} ,$$

respectively. From Eq.(8) the necessary condition for $J(\omega)$ to be minimum is also paraphrased in terms of approximate nodal values in the form $\{\boldsymbol{r}_2\} = \{\boldsymbol{0}\}$.

We substitute all of these boundary nodal forms into linear systems Eqs.(15) and (16), and combine them into a large linear system of equations of order $2n$ with unknown nodal vectors $\{\boldsymbol{q}_1\}$, $\{\boldsymbol{\omega}\}$, $\{\boldsymbol{r}_1\}$, and $\{\boldsymbol{q}_2\}$. The resulting linear system can be written in the form

$$\begin{array}{c} \\ n \\ n \end{array} \begin{array}{c} \begin{array}{cccc} n_2^c & n_1^c & n_1 & n_2 \end{array} \\ \begin{pmatrix} -\boldsymbol{G}_1^{(1)} & \boldsymbol{H}_2^{(1)} & \boldsymbol{O} & -\boldsymbol{G}_2^{(1)} \\ 2\eta\boldsymbol{H}_1^{(2)} & \boldsymbol{O} & -\boldsymbol{G}_2^{(2)} & 2(1+\eta)\boldsymbol{H}_2^{(2)} \end{pmatrix} \end{array} \begin{Bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{\omega} \\ \boldsymbol{r}_1 \\ \boldsymbol{q}_2 \end{Bmatrix}$$

$$= \begin{array}{c} \\ n \\ n \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} -\boldsymbol{H}_1^{(1)} & \boldsymbol{O} \\ \boldsymbol{O} & 2\boldsymbol{H}_2^{(2)} \end{pmatrix} \end{array} \begin{Bmatrix} \bar{\boldsymbol{u}}_1 \\ \bar{\boldsymbol{q}}_2 \end{Bmatrix} . \tag{17}$$

It is expected that a suitable value of the regularization parameter η is determined by the compromise between the estimation of observation errors contained in data $\bar{u}$ and $\bar{q}$ and the accuracy of boundary element approximation. But in practice we choose the value by Hansen's L–curve method [4].

3.2 High order finite difference method

We take a point $\boldsymbol{x} = (x_1, x_2)$ and N distinct points $\boldsymbol{x}^{(j)} = (x_1^{(j)}, x_2^{(j)})$ in $\bar{\Omega} = \Omega \cup \Gamma$ for $j = 0, 1, \cdots, N-1$. These points are called quadrature points. We also take vectors $\boldsymbol{h} = (h_1, h_2)$ and $\boldsymbol{h}^{(j)} = (h_1^{(j)}, h_2^{(j)})$ for $j = 0, 1, \cdots, N-1$ so that $\boldsymbol{x}^{(j)} = \boldsymbol{x} + \boldsymbol{h}^{(j)} \in \bar{\Omega}$. For the sake of convenience, we express the Laplacian by

$$P(\boldsymbol{\partial}) = \boldsymbol{\partial}^{(2,0)} + \boldsymbol{\partial}^{(0,2)} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} ,$$

where $\boldsymbol{\partial} = (\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}) = \nabla$.

For unknown function $u(\boldsymbol{x})$, we approximate $P(\boldsymbol{\partial})u(\boldsymbol{x}+\boldsymbol{h})$ by a linear combination of $u(\boldsymbol{x}^{(j)})$ of the form

$$P(\boldsymbol{\partial})u(\boldsymbol{x}+\boldsymbol{h}) \approx \sum_{j=0}^{N-1} w_j(\boldsymbol{h})u(\boldsymbol{x}^{(j)}) \tag{18}$$

with the weights $w_j(\boldsymbol{h})$. For large N we call the approximation Eq.(18) high order finite difference approximation.

Heat Transfer VIII, B. Sundén, C. A. Brebbia & A. Mendes (Editors)
© 2004 WIT Press, www.witpress.com, ISBN 1-85312-705-1

Now we try to determine the weights by applying the Fourier transform to the unknown function $u(\boldsymbol{x})$. To this end, we extend the domain Ω to the whole plane $\boldsymbol{R}^2$, and we assume that thus extended $u(\boldsymbol{x})$ is a tempered distribution. Then the Fourier transform of $u(\boldsymbol{x})$ is defined by $\langle F[u], \varphi \rangle = \langle u, F[\varphi] \rangle \quad \text{for } \forall \varphi \in S$, where S is a set of all rapidly decreasing functions on $\boldsymbol{R}^2$, $\langle \cdot, \cdot \rangle$ indicates the dual pairing between the space S and its dual space S', and $F[\varphi](\boldsymbol{\xi}) = \int_{\boldsymbol{R}^2} \varphi(\boldsymbol{x}) e^{-\sqrt{-1}\boldsymbol{\xi}\cdot\boldsymbol{x}} d\boldsymbol{x}\,, \quad \boldsymbol{\xi} \in \boldsymbol{R}^2$.

We notice that

$$F[P(\boldsymbol{\partial})u](\boldsymbol{\xi}) = P(\sqrt{-1}\boldsymbol{\xi})F[u](\boldsymbol{\xi})\,, \quad F[u(\cdot + \boldsymbol{h})](\boldsymbol{\xi}) = e^{\sqrt{-1}\boldsymbol{\xi}\cdot\boldsymbol{h}} F[u](\boldsymbol{\xi})\,.$$

The left and right hand sides of Eq.(18) are transformed, respectively, on the Fourier domain into

$$F[P(\boldsymbol{\partial})u(\cdot + \boldsymbol{h})](\boldsymbol{\xi}) = P(\sqrt{-1}\boldsymbol{\xi}) e^{\sqrt{-1}\boldsymbol{\xi}\cdot\boldsymbol{h}} F[u](\boldsymbol{\xi})\,, \tag{19}$$

$$F\left[\sum_{j=0}^{N-1} w_j(\boldsymbol{h}) u(\cdot + \boldsymbol{h}^{(j)})\right](\boldsymbol{\xi}) = \sum_{j=0}^{N-1} w_j(\boldsymbol{h}) e^{\sqrt{-1}\boldsymbol{\xi}\cdot\boldsymbol{h}^{(j)}} F[u](\boldsymbol{\xi})\,. \tag{20}$$

When we equate these two expressions, we have

$$\left\{ P(\sqrt{-1}\boldsymbol{\xi}) - \sum_{j=0}^{N-1} w_j(\boldsymbol{h}) e^{\sqrt{-1}\boldsymbol{\xi}\cdot(\boldsymbol{h}^{(j)} - \boldsymbol{h})} \right\} F[u](\boldsymbol{\xi}) = 0\,. \tag{21}$$

We can replace the factor $\sqrt{-1}\boldsymbol{\xi}$ in this expression by $\rho(\boldsymbol{h}^{(i)} - \boldsymbol{h})$ with a real parameter $\rho > 0$. Then we require that the resulting equality holds for all $i = 0, 1, \cdots, N-1$ and for all $F[u]$ to obtain

$$P(\rho(\boldsymbol{h}^{(i)} - \boldsymbol{h})) = \sum_{j=0}^{N-1} w_j(\boldsymbol{h}) e^{\rho(\boldsymbol{h}^{(i)} - \boldsymbol{h})\cdot(\boldsymbol{h}^{(j)} - \boldsymbol{h})}\,. \tag{22}$$

We construct $N \times N$ matrix $\boldsymbol{L}_{\boldsymbol{h}} = (\ell_{ij}(\boldsymbol{h}))$ with the components $\ell_{ij}(\boldsymbol{h}) = e^{\rho(\boldsymbol{h}^{(i)} - \boldsymbol{h})\cdot(\boldsymbol{h}^{(j)} - \boldsymbol{h})}$, $N-$column vector $\boldsymbol{w}_{\boldsymbol{h}} = (w_j(\boldsymbol{h}))$, and $N-$column vector $\boldsymbol{b}_{\boldsymbol{h}} = (b_i(\boldsymbol{h}))$ with components $b_i(\boldsymbol{h}) = P\left(\rho(\boldsymbol{h}^{(i)} - \boldsymbol{h})\right)$. Then the equation Eq.(22) is the system of linear equations $\boldsymbol{L}_{\boldsymbol{h}}\boldsymbol{w}_{\boldsymbol{h}} = \boldsymbol{b}_{\boldsymbol{h}}$. Accordingly, the weights $w_j(\boldsymbol{h})$ can be given as the solution $\boldsymbol{w}_{\boldsymbol{h}} = \boldsymbol{L}_{\boldsymbol{h}}^{-1}\boldsymbol{b}_{\boldsymbol{h}}$ only when $\boldsymbol{L}_{\boldsymbol{h}}$ is non-singular.

We recall that the Dirichlet data $\bar{u}(\boldsymbol{x})$ are imposed on Γ_u and the Neumann data $\bar{q}(\boldsymbol{x})$ are imposed on Γ_q. Among the N quadrature points $\boldsymbol{x}^{(k)}$ for $k = 0, 1, \cdots, N-1$, let $I_\Omega = \{0, 1, \cdots, N_0 - 1\}$ is the index set for which the quadrature points $\boldsymbol{x}^{(k)} \in \Omega$ when $k \in I_\Omega$, let $I_u = \{N_0, N_0 + 1, \cdots, N_1 - 1\}$ is the index set for which the quadrature points $\boldsymbol{x}^{(k)} \in \Gamma_u$ when $k \in I_u$, and let $I_q = \{N_1, N_1 + 1, \cdots, N - 1\}$ is the index set for which the quadrature points $\boldsymbol{x}^{(k)} \in \Gamma_q$ when $k \in I_q$.

Heat Transfer VIII, B. Sundén, C. A. Brebbia & A. Mendes (Editors)
© 2004 WIT Press, www.witpress.com, ISBN 1-85312-705-1

We define a differential operator $P_k(\boldsymbol{\partial})$ and data f_k by

$$P_k(\boldsymbol{\partial}) = \begin{cases} P(\boldsymbol{\partial}) \\ Id \\ \boldsymbol{n}(\boldsymbol{x}^{(k)}) \cdot \nabla \end{cases} \qquad f_k = \begin{cases} 0, & k \in I_\Omega \\ \bar{u}(\boldsymbol{x}^{(k)}), & k \in I_u \\ \bar{q}(\boldsymbol{x}^{(k)}), & k \in I_q \end{cases}$$

Here Id is the identity operator, and $\boldsymbol{n}(\boldsymbol{x}^{(k)})$ is the unit outward normal at the point $\boldsymbol{x}^{(k)}$. Then the Cauchy problem Eqs.(1) and (3) is considered only at the quadrature points as follows:

$$P_k(\boldsymbol{\partial})u(\boldsymbol{x}^{(k)}) = f_k\,, \quad k = 0, 1, \cdots, N-1\,. \tag{23}$$

From Eq.(18) we can approximate the left hand side of this equation by

$$P_k(\boldsymbol{\partial})u(\boldsymbol{x}^{(k)}) \approx \sum_{j=0}^{N-1} w_{kj} u(\boldsymbol{x}^{(j)})\,. \tag{24}$$

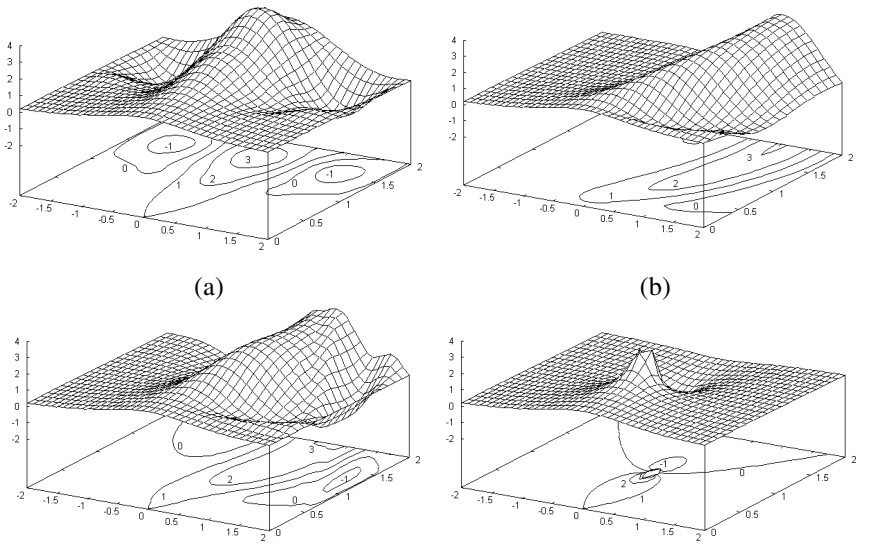
Then the k-th weight vector $\boldsymbol{w}^{(k)} = (w_{kj})$ is determined from the system of linear equations $\boldsymbol{L}_k \boldsymbol{w}^{(k)} = \boldsymbol{b}^{(k)}$ with the components $\ell_{ij}^{(k)} = e^{\rho(\boldsymbol{h}^{(i)} - \boldsymbol{h}^{(k)}) \cdot (\boldsymbol{h}^{(j)} - \boldsymbol{h}^{(k)})}$ and $b_i^{(k)} = P_k\left(\rho(\boldsymbol{h}^{(i)} - \boldsymbol{h}^{(k)})\right)$. From Eqs.(23) and (24), the approximate $u-$values u_j to the exact $u(\boldsymbol{x}^{(j)})$ can be obtained from the system of linear equations $\sum_{j=0}^{N-1} w_{kj} u_j = f_k\,, \quad k = 0, 1, \cdots, N-1\,.$

4 Calculated results

Surface and its contours of the exact solution given by Eq.(2) are presented in Figure 2(d) on the domain $\Omega = (-2, 2) \times (0, 2)$. We can see that the singularity at the point $(0, 1)$ is a doublet. Figure 2(a) shows boundary element solutions of $u(x_1, a)$ for $-2 \le x_1 \le 2$ and $0 < a \le 2$ with the increment $\Delta a = 0.1$ for Problem [A]. We can see that the curve $u(x_1, a)$ is uni-modal for $a < \beta = 1$ and it becomes multi-modal for $a > \beta$. Figure 2(b) shows boundary element solutions of $u(b, x_2)$ for $-2 < b \le 2$ and $0 \le x_2 \le 2$ with the increment $\Delta b = 0.1$ for Problem [B]. We can see that the curve $u(b, x_2)$ begins to vary when b becomes larger than $\alpha = 0$. Figure 2(c) shows boundary element solutions of $u(x_1, x_2)$ for $-2 < c \le 4$ with the increment $\Delta c = 0.1$. We can see that the curve $u(x_1, x_2)$ behaves oddly as the value of c passes across $\beta = 1$. We notice that the boundary element solutions presented in (a)-(c) of Figure 2 differs significantly from each other, although they stem essentially from the same Cauchy problem. A cause for the differences is thought due to existence of the singular point.

Figure 3(a) shows the difference in modulus between surfaces of (a) and (b) in Figure 2. We notice small differences in the area $\{\ (x_1, x_2)\ |\ -2 \le x_1 < 0,\ 0 \le$

Heat Transfer VIII, B. Sundén, C. A. Brebbia & A. Mendes (Editors)
© 2004 WIT Press, www.witpress.com, ISBN 1-85312-705-1

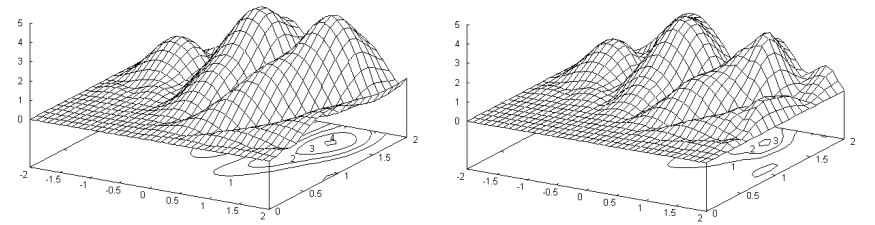


(a) (b)

(c) (d)

Figure 2: The exact and boundary element solutions.

$x_2 < 1$ } and large differences on the rest in Ω. Figure 3(b) shows the difference in modulus between surfaces of (a) and (c) in Figure 2. We notice small differences in the area $\{ (x_1, x_2) \mid x_2 < -x_1 + 1 \}$ and large differences on the rest. As a consequence of our scans by using three linearly independent family of lines $x_2 = a$, $x_1 = b$, and $x_2 = -x_1 + c$, we detect an isolated essential singular point (α, β) near the location $(0, 1)$.



(a) (b)

Figure 3: Differences in the boundary element solutions.

Figures 4 (a) and (b) show comparison between contours of the exact solution and those of high order finite difference solutions when the number of quadrature

Heat Transfer VIII, B. Sundén, C. A. Brebbia & A. Mendes (Editors)
© 2004 WIT Press, www.witpress.com, ISBN 1-85312-705-1

points is set as $N = 100$ and $N = 200$, respectively, for Problem [A] with $a = 2$. We can see that the contours of the finite difference solutions agree well with those of the exact solution only in vicinity to the initial line $x_2 = 0$. When the value N is varied from 100 to 200, the numerical results change drastically, particularly in the area beyond the singular point. As we increase the value of N, numerical solutions do not converge because a singular point exists inside the domain.

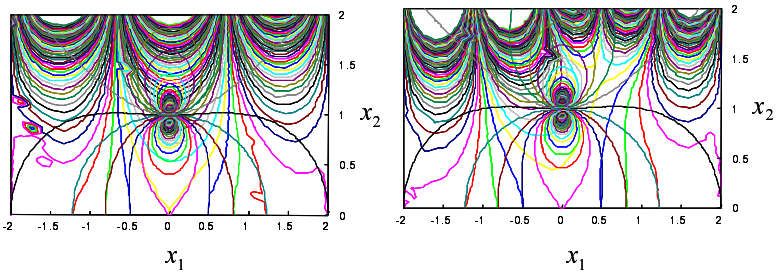


Figure 4: Contours of the exact and high order finite difference solutions (a) $N = 100,\ \rho = 1$, (b) $N = 200,\ \rho = 1$.

5 Conclusions

A detection method is proposed for an isolated essential singular point of a harmonic function inside the domain. Dirichlet data and Neumann data are assumed to be available on a respective part of the boundary. Cauchy problems of the Laplace equation are considered for a scanning process by using a series of subdomains. We try to see if the data can be extended to the other side of the initial curve where the Dirichlet and Neumann data are prescribed for each subdomain. When some symptom appears, indicating difficulties in the extension, we conceive that a singular point is met with at that scanning process. The Cauchy problem is numerically solved by either the boundary element method or high order finite difference method. Only one sample problem is examined. Using three linearly independent scanning lines we succeeded in detecting a singular point of the harmonic function in the sample problem.

References

[1] D. Vvedensky, *Partial Differential Equations with Mathematica*^{R}, Addison–Wesley Publishers Ltd., (1993).

[2] K. Iijima, Numerical method for backward heat conduction problems using an arbitrary–order finite difference method, *Proceedings of the International*

© 2004 WIT Press, www.witpress.com, ISBN 1-85312-705-1

Symposium on Inverse Problems in Engineering Mechanics 2003, Nagano, Japan (in print).

[3] K. Hayashi, Y. Ohura, and K. Onishi, Direct method of solution for general boundary value problem of the Laplace equation, *Engineering Analysis with Boundary Elements*, Vol. 26, (2002), pp. 763-771.

[4] P. C. Hansen, Analysis of discrete ill–posed problems by means of the $L-$curve, *SIAM Review*, Vol. 34, No. 4, (1992), pp. 561-580.



Heat Transfer VIII, B. Sundén, C. A. Brebbia & A. Mendes (Editors)
© 2004 WIT Press, www.witpress.com, ISBN 1-85312-705-1