Mass and thermal flow in glaciers

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Abstract

Glaciers play a fundamental role in the study of pollution and climate changing; from ice cores it is in fact possible to track the evolution of chemical and physical properties of the atmosphere over different scales of time and space.

A temperature vs. depth profile is required for drilling operations while an age vs. depth profile is required for data interpretation. Such information comes from the knowledge of the thermo-dynamics of the glacier under study and thus, from a good mathematical flow model.

Ice thermo-dynamics is non-linear and convection dominated. Both these characteristics represent the major tasks to be faced during the formulation of a numerical model. A number of models have been proposed before, but none seems to be efficient enough to model a section of complex geometry such as sections of Alpine glaciers. The mass-flow model proposed in this work is based on the assumptions that the pressure is lithostatic throughout the domain and that stress and velocity are twice differentiable vector functions. Under these assumptions it is possible to split the entire system into smaller and simpler systems and to double derive the equations in such a way that second order terms are obtained. The velocity field obtained is the input for computing the heat transfer equation. All the systems are solved by the Finite Difference Method using a regular but non-uniform grid.

1 Introduction

Many studies on global environmental and climatic variations are based on analyses of ice cores extracted by means of deep drilling in Polar Regions. In these remote sites the sources of pollution are very low; it is thus possible to gain information on globally averaged values of the chemical and physical composition of the atmosphere. Middle latitude glaciers, on the contrary, are
often located in regions with relevant anthropogenic development. Ice cores extracted from these sites can thus provide information about the direct anthropogenic impact on the atmosphere (Wagenbach [8]). Drilling operations and data interpretation call for a precise knowledge of ice thermodynamics. In this scenario mathematical modelling represents a powerful tool for glaciological studies.

A number of mathematical flow models have been developed for ice caps and ice sheet, but most of the available ones do not seem to be very effective on valley glacier sections. We thus aim to develop a mathematical model capable to describe the thermodynamics of sections of ice masses of arbitrary geometry. Since this is a formidable task to be achieved, we started working on a simple parallel sided section.

2 The physical problem

Ice mechanics is described by the fundamental laws of conservation of mass, linear and angular momentum. These laws can be written in differential form and set relations between the stress field \((\sigma_{ij}, i,j=x,y,z)\) and the velocity field \((v)\):

\[
\begin{align*}
\nabla \cdot v &= 0 \quad \text{[mass cons.]} \\
\frac{\partial \sigma_{xi}}{\partial x} + \frac{\partial \sigma_{yi}}{\partial y} + \frac{\partial \sigma_{zi}}{\partial z} + \rho g_i &= 0 \quad i = x, y, z \quad \text{[lin. mom. cons.]} \\
\sigma_{ij} - \sigma_{ji} &= 0 \quad i, j = x, y, z \quad \text{[ang. mom. cons.]} 
\end{align*}
\]

where \(\rho\) is the ice density and \(g_i\) is the component of the gravity acceleration vector along the \(x_i\) coordinate.

These laws hold for any continuous medium and, alone, are not able to describe the physical properties of ice. Moreover, in the system obtained by the combination of these laws, the number of unknowns is greater than the number of equations. A flow law is thus required both for completing the system and for describing ice rheology. It will thus set a relation between the stress field and the velocity field; the general form of a flow law is:

\[
\frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \lambda \sigma'_{ij} \quad i, j = x, y, z ,
\]

where:
- \(u_i\) is the velocity component along the \(x_i\) coordinate;
- \(\lambda\) is a function dependent on the stress field that describes ice rheology;
- \(\sigma'_{ij}\) is the component of the deviatoric stress tensor obtained by the formula:

\[
\sigma'_{ij} = \sigma_{ij} + \delta_{ij} p
\]

where \(p\) is the pressure.
If ice is considered as a perfectly plastic material, then the stress acting throughout the glacier will be constant and will equal a critical value \( k \). In this case the flow law is linear. In reality ice may be considered perfectly plastic only in the lower half, where high pressure occurs. A better description of ice rheology could be achieved by considering it as a visco-plastic material. In this case \( \lambda \) can be expressed (as suggested by Glen in 1952) by:

\[
\lambda = A \sigma_e^{n-1},
\]

where:
- \( A \) is a flow parameter that depends on temperature, ice crystal size and orientation, impurity content and creep activation energy (Patterson [6]).
- \( n \) is another flow parameter that can vary form 1 to 6 but is normally set to 3 (Hobbs [3]).
- \( \sigma_e \) is the effective stress and does not depend on the axis orientation. The relation between effective and deviatoric stresses is:

\[
\sigma_e = \sqrt{\frac{1}{2} \left( \sigma'_x + \sigma'_y \right) + \sigma'_z}.
\]

It can be seen that the system obtained by combining the conservation laws and the flow law is dominated by first order derivatives that may lead to numerical instability when numerical methods are used to compute the solution. Moreover, if ice is considered visco-plastic, the entire system of simultaneous equations is non-linear and the heat-flow equation must be considered since the flow parameter \( A \) is temperature dependent. The general heat-flow equation is:

\[
\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta + \nabla \cdot \kappa \cdot \nabla \theta - \mathbf{v} \cdot \nabla \theta + \frac{Q}{\rho C},
\]

where:
- \( \theta \) is the temperature.
- \( \kappa \) is the thermal diffusivity.
- \( Q \) is the heat production term due to stress and strain.
- \( C \) is the ice thermal capacity.

The thermal diffusivity \( \kappa \) is temperature dependent; since the temperature varies in space, also \( \kappa \) varies in space. Nevertheless the thermal diffusivity of ice is 37 m\(^2\)a\(^{-1}\) at 0°C and 58 m\(^2\)a\(^{-1}\) at -60°C. This small variation of \( \kappa \), in combination with small temperature gradients, makes the term \( \nabla \kappa \cdot \nabla \theta \) negligible in comparison with the others in eqn (6) (see also Hooke [4]). Moreover, most glaciers and ice caps can be assumed to be in a steady state, so that we can impose \( \partial \theta / \partial t = 0 \). The heat-flow equation can thus be rewritten as:

\[
\kappa \nabla^2 \theta - \mathbf{v} \cdot \nabla \theta + \frac{Q}{\rho C} = 0.
\]
It can be noticed that the temperature distribution in a glacier is influenced by
the stress and strain fields through the source term $Q/\rho C$ and by the velocity field
through the advective term $\mathbf{v} \cdot \nabla \theta$. Once more this demonstrates how a close
coupling between mass flow and thermal flow models is required in order to
precisely describe ice thermodynamics. Of course this is the target toward which
present and future research has to aim, but at the moment it seems to be a
formidable task to be achieved. For this reason our work started by solving the
heat-flow equation and the system describing ice mechanics separately. The
coupling between these two models will be the object of future research.

3 The parallel sided section

As shown in the previous section, the convective nature and the non-linearity of
the problem represent two big problems to be faced in solving the system. If, to
these, we add a complex geometry such as the one of a typical Alpine glacier
section, it can be understood how difficult the problem gets. This is why we
consider a simple parallel sided section as a starting point of this research. If we
also consider a bi-dimensional reference system in which the x-axis is parallel to
the bedrock and the z-axis is directed upward with the origin on the bedrock,
some simplifications in the calculations can be made and the system can be
solved. If we consider the flow to be laminar (Nye [5]), i.e. the flow-lines are
always parallel to surface and bedrock, then the absolute value of each stress
component linearly increases with depth, while velocities decrease with depth
with the profile typical for a visco-plastic material. The stress field is described
by the stress tensor:

$$\sigma = -\rho g (h - z) \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix},$$

where:
- $g$ is the gravity acceleration;
- $h$ is the surface z-coordinate;
- $\alpha$ is the inclination of the surface.

The velocity field is described by the velocity vector:

$$\mathbf{v} = \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} u_0 + \frac{2A}{n+1} (-\rho g \sin \alpha)^n [(h)^{n+1} - (h - z)^{n+1}] \\ 0 \end{bmatrix},$$

where $u_0$ is the sliding velocity.

The laminar flow solution is interesting but not very realistic. In order to have
a more realistic one, we should consider the effect of accumulation or ablation of
mass (Hooke [4]). In this case, at first, it is convenient to consider ice as a
perfectly plastic material. Therefore, the total stress is constant throughout the
domain and equals the critical stress \( k \). It can be shown that the increase in stress components is linear except for the normal stress in the \( x \)-direction that is non-linear. The components of the stress tensor are:

\[
\sigma_{xz} = k \left( 1 - \frac{z}{h} \right), \\
\sigma_{xx} = -\rho g (h-z) \pm 2k \sqrt{1 - \left( 1 - \frac{z}{h} \right)^2}, \\
\sigma_{zz} = -\rho g (h-z),
\]

where in eqn (11) the plus sign is used for the ablation area and the minus sign for the accumulation area.

If the effect of accumulation/ablation is considered, then the flow-lines are submerging in the accumulation area and emerging in the ablation area. This effect is described by the solution for the velocity components:

\[
u = \frac{b_n}{h} x + 2 |b_n| \sqrt{1 - \left( 1 - \frac{z}{h} \right)^2} + c, \\
w = -\frac{b_n}{h} z,
\]

where \( b_n \) is the net balance of accumulation/ablation and \( c \) is an integration constant depending on boundary conditions.

We use the latter velocity solution in order to calculate a temperature vs. depth profile for the parallel sided section considered. In order to integrate the heat-flow equation and thus find a solution, we consider the source term to be negligible. The integration of the second order equation of heat-flow brings two constants: one is the surface temperature, the other is the temperature gradient at the bedrock. This choice can easily be explained: the surface temperature can be measured on site while the temperature gradient at the bedrock can be calculated by dividing the typical value of geothermal flux by ice density (\( \rho \)) and ice thermal capacity (\( C \)). If we consider a section few hundred meters long and slightly tilted, then the surface temperature has to be constant and independent of \( x \). Also the temperature gradient on the bedrock is constant and independent of \( x \). For these reasons the temperature can be considered independent of \( x \) throughout the domain. The solution for the temperature vs. depth profile is:
\[ \theta(z) = \theta(h) - \frac{\sqrt{\pi}}{2} \frac{\beta_0}{\xi} [\text{erf}(\xi h) - \text{erf}(\xi z)] , \]  \hspace{1cm} (15)

where:
- \( \beta_0 \) is the geothermal flux divided by \( \rho C \);
- \( \xi = \sqrt{b_n / 2 \kappa h} \);
- \( \text{erf} \) is the error function and is defined as:

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt . \]  \hspace{1cm} (16)

4 The heat-flow model

In the previous sections we briefly outlined the background knowledge needed to develop a heat-flow model for a glacier section. As said above, we start by considering a parallel sided section with a reference system in which the \( x \)-axis is parallel to the bedrock. We impose the velocity field using eqns (13) and (14) and we consider the source term to be negligible.

4.1 The parameters

A section 200 m long and 120 m deep is considered; the section is discretized by using a 5 m pace both in the \( x \)- and \( z \)-direction. On this regular scheme the Finite Difference Method is applied; all the lengths are expressed in km. A 10\% inclination is considered for surface and bedrock. The section is considered to be in the accumulation area and the net balance to be of 70 cm per year. The critical stress \( k \) is set to 100 kPa. The surface temperature is considered to be -10°C; the calculated basal ice temperature variation along the \( z \)-coordinate is -0.226 K per meter. The thermal diffusivity \( \kappa \) is set to 40 m\(^2\) per year.

4.2 Dirichlet boundary conditions

The first application is governed by Dirichlet boundary conditions on all the boundaries; the value of the temperature at the boundary is calculated using eqn (15). In this case a central difference scheme of second order is used to discretize the equations. The result is shown in Figure 1 and is in perfect agreement with the theoretical one: the contours are always parallel to surface and bedrock, meaning that the effect of the temperature gradient in the \( x \)-direction is negligible, as expected. Moreover the temperature vs. depth profile is non-linear in agreement with eqn (15).
4.3 Mixed Dirichlet-Neumann boundary conditions

As a first step we impose Neumann boundary conditions only on the bedrock; in particular we impose the thermal vertical variation to equal $\beta_0$ as we did for eqn (15). In this case the system is solved also for the points lying on the bedrock; here, however, the heat-flow equation is not imposed, but only the condition on the normal derivative is. For the points on the bedrock an upwind non-symmetrical scheme of second order is used. The result (not shown here) is identical to the one obtained by imposing Dirichlet conditions. Subsequently Neumann boundary conditions are imposed also on the lateral boundaries; in this case we impose the horizontal gradient to be null (as happens on the surface). Again, the system includes equations also for the points on the lateral boundaries but here the heat-flow equation is not imposed. Downwind (for left boundary) and upwind (for right boundary) non-symmetrical stencils are used. The result is shown in Figure 2; it shows the expected behaviour and accuracy. Again on the left, right and lower boundaries the heat-flow equation is not imposed but only the condition on the normal derivative is.

5 The mass-flow model

We are now interested in formulating a mass-flow model for a parallel sided section. In order to give generality to this model we are going to consider a coordinate system in which the $x$-axis is tangent to the earth surface and the $z$-axis is perpendicular to the earth surface. The origin is chosen in such a way that the lowest point of the section has null $z$-coordinate. This coordinate system will facilitate the application of the model and of the solving algorithm to sections of general geometry.

Using eqn. (3) it is possible to rewrite system (1) in terms of deviatoric stresses and pressure. At first, we assume that the flow regime in the section is laminar; this implies that the pressure can be considered purely lithostatic throughout the domain. This assumption permits to split the system into two smaller and simpler systems: one for the deviatoric stress field and the other for the velocity field (for a more detailed treatment refer to Deponti [2]). Assuming that the stresses and the velocities are twice differentiable vector functions, then it is possible to differentiate each equation with respect to $x$ and $z$. After some simple calculations, the following second order systems of more straightforward solution are obtained:

$$
\begin{align*}
\nabla^2 (\sigma'_{xx}) &= 0 \\
\nabla^2 (\sigma'_{xz}) &= 0,
\end{align*}
$$

$$
\begin{align*}
\nabla^2 u &= 2 \frac{\partial}{\partial x} \left( A \sigma_e^{n-1} \sigma'_{xx} \right) + 2 \frac{\partial}{\partial z} \left( A \sigma_e^{n-1} \sigma'_{xz} \right) \\
\nabla^2 w &= 2 \frac{\partial}{\partial x} \left( A \sigma_e^{n-1} \sigma'_{xz} \right) - 2 \frac{\partial}{\partial z} \left( A \sigma_e^{n-1} \sigma'_{xx} \right),
\end{align*}
$$

(17)
Figure 3: Stress field.

Figure 4: Velocity field.
5.1 The numerical solution

In order to numerically solve eqns (17) and (18) the Finite Difference Method is used. Central difference schemes of second order are used throughout the domain, but, since the upper and lower boundaries are tilted with respect to grid lines, non-uniform stencils occur near surface and bedrock.

Dirichlet boundary conditions are applied to all the boundaries using eqns (8) and (9). The results are shown in figure 3 and 4. In Figure 3 the values of the effective stress calculated by eqn (5) are plotted, while in Figure 4 the velocity vectors are plotted. It can be noticed that the stress field linearly increases with depth, while the velocity decreases non-linearly with depth and the velocity vectors are parallel to surface and bedrock, as expected.

6 Conclusions and future works

The most important issues concerning the equations that describe ice thermodynamics are non-linearity and dominance of convective terms. When one is to solve these equations in real instances, the complex geometry of glacier sections and the difficulties in identifying good boundary conditions make the problem awkward. With this work we demonstrate that it is possible to solve the problem by making some reasonable simplifying assumptions. The target of future research is to leave these simplifying assumptions one by one in order to model ice thermodynamics in general cases.

References