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Numerical study of laminar natural convection of air in an annulus allowing for variable fluid properties

F. Shahraki & R. Bragg
Department of Chemical Engineering,
UMIST, Manchester M60 1QD, UK

Abstract

In this paper the effects of variable thermo-physical properties of the fluid on laminar natural convection heat transfer and fluid flow were determined using a penalty finite element method for an annulus between two concentric cylinders problem with radius ratio 2.6 (R_{out}/R_{in}=2.6), considering both constant and variable fluid properties. The penalty finite element method was used for Rayleigh numbers ranging from 10^3 to 10^5 and $Pr_0 = 0.71$. The formulation was based on primitive variables.

1 Introduction

The well-known Boussinesq approximation has been extensively used but very little research has been done to check the influence of variable thermo-physical properties in the flow structure with or without the Boussinesq simplification. The full Boussinesq approximation is limited to moderate temperature differences. The next stage of approximation is to modify the method to allow for temperature-dependent viscosity and thermal conductivity. The variation of fluid density is still restricted to the buoyancy term.

In natural convection flows, the fluid velocities are extremely dependent on the temperature field and variable properties that can have strong effects on both the velocity components and the Nusselt number [1]. The basic effects of the properties are in the dimensionless groups, the Rayleigh and Prandtl numbers. It is common to assume that for temperature-dependent properties, the viscosity and thermal conductivity vary in such a way that the

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Prandtl number remains constant or that average properties can be used in evaluating Prandtl and Rayleigh numbers.

The effect of the variation of viscosity with temperature has been studied by Yamasaki and Irvine [2] and Hyun and Lee [3]. However, the effect of temperature-dependent thermal conductivity, which can significantly affect the thermal field, has received little attention, particularly in conjunction with a simultaneous variation in the viscosity. Chenoweth and Paolucci [4] studied the natural convection of an ideal gas where both the thermal conductivity and viscosity followed Sutherland's law, but only for $Pr_0 = 0.71$. Emery and Lee [1] conducted a comprehensive set of temperature-dependent conductivity and viscosity experiments over a wide range of Prandtl number. They presented the results for $0.01 \leq Pr \leq 1$, which is representative of liquid metals and gases. All other properties (β, ρ and c_p) were taken as constant.

In this paper, a modified Boussinesq approximation with temperature-dependent viscosity and thermal conductivity has been used. The effects of temperature-dependent fluid properties on laminar natural convection heat transfer and fluid flow were determined using a penalty finite element method for the classical steady, two-dimensional, differentially heated, square cavity problem and concentric pipes, considering both, viscosity and thermal conductivity to vary with temperature.

2 Governing equations

The problem under consideration is governed by steady, two-dimensional equations of continuity, momentum and energy. There are a variety of physical constants in the basic equations such as density, viscosity, thermal conductivity, heat conduction coefficient, etc. The values of these constants generally depend on the temperature. There is a physical assumption, called the Boussinesq approximation used in this work. For accurate calculation of the overall heat transfer rate, the limit of validity of the Boussinesq approximation has been recommended [4] as

$$\frac{T_{hot} - T_{cold}}{T_{hot} + T_{cold}} < 0.6 \qquad (1)$$

However, for accurate determination of velocity and temperature fields this temperature ratio should be limited to an upper value of 0.2 [4]. In order to demonstrate the effects of temperature-dependent physical properties for natural convection, a modified Boussinesq approximation with temperature-dependent viscosity and thermal conductivity has been used (the density is also considered constant except in the buoyancy term). In addition, the heat generation due to viscous dissipation is assumed to be negligible.

It is well known that formulation based on primitive variables involves difficulties with the treatment of the pressure and continuity equation. In order to overcome these difficulties, a penalty function formulation has been adopted

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in this study. The penalty function approach has been widely used for the prediction of incompressible flows, especially in the context of finite element techniques such as those used by Hughes et al. [5], Reddy [6] and Syrjala [7].

The penalty formulation has attracted considerable interest of investigators because it eliminates the pressure from the momentum equations by using a pseudo-constitutive equation to express the pressure in terms of the fluid dilation. This then significantly reduces the number of degrees of freedom. In addition, the calculated pressures never influence the velocity field because incompressibility (or approximate incompressibility) is imposed directly by restricting the space of acceptable trial functions through the penalty term. Therefore, inaccuracies in the pressure do not affect convergence of the velocity field although, for the most generally used elements, pressure does not even converge if calculated from the pseudo-constitutive relation introduced in the penalty method.

Use of the penalty function formulation amounts to replacing the continuity requirement by the weakened constraint

$$P = -\lambda\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) \tag{2}$$

where λ is the penalty parameter. As stressed by Reddy [8], Eqn (2) does not define the penalty method. The method consists of solving the momentum and energy equations subject to the constraint of satisfying the continuity equation. The equation of continuity is no longer satisfied exactly when the penalty function method is used. However, a sufficiently large value of λ, typically 10^5-10^{12} (the penalty parameter should be chosen to be several orders of magnitude greater than the viscosity) ensures that Eqn (2) is almost equivalent to the original continuity equation.

The dimensionless quantities can be expressed as

$$\begin{gathered}
X = \frac{x}{L} \qquad Y = \frac{y}{L} \qquad U = \frac{uL}{(k/\rho C_P)} \qquad V = \frac{vL}{(k/\rho C_P)} \\
P = \frac{pL^2}{\rho(k/\rho C_P)^2} \qquad \theta = \frac{T - T_c}{T_h - T_c} \\
\Pr = \frac{(\mu/\rho)}{(k/\rho C_P)} \qquad \mathrm{Ra} = \frac{g\beta(T_h - T_c)L^3}{(k/\rho C_P)(\mu/\rho)} \\
f_1 = \frac{\mu}{\mu_0} = F_1(\theta) \qquad f_2 = \frac{k}{k_0} = F_2(\theta)
\end{gathered} \tag{3}$$

where L is the reference length. The technique has been extended to use property factors f_1 and f_2 given in the momentum and energy equations. For an initial comparison, following Emery and Lee [1], the two factors were set to

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$$f_1 = f_2 = 0.5 + 0.5\theta + \theta^2 \tag{4}$$

Consequently, both factors have the value 1.0 at the average temperature, 0.5 at the cold surface and 2.0 at the hot surface. These very large variations of properties have been selected to demonstrate the relative sensitivities of the model to changes in these two properties, all other properties being assumed constant. It must be stressed that these conditions are artificial: the temperature differences that would be required are enormous and, in the case of air, the density, in particular, would change greatly. The governing partial differential equations can be written in dimensionless form as follows.

When Eqn (2) is then used to substitute for the pressure, the continuity equation is eliminated from the preceding system of equations, and the final equations to be solved become:

$$-2\mathrm{Pr}_0 \frac{\partial}{\partial X}\left[f_1 \frac{\partial U}{\partial X}\right] - \mathrm{Pr}_0 \frac{\partial}{\partial Y}\left[\left(f_1 \frac{\partial U}{\partial Y} + f_1 \frac{\partial V}{\partial X}\right)\right] - \lambda \frac{\partial}{\partial X}\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) + \frac{\partial U}{\partial t^*} + \left[U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y}\right] = 0 , \tag{5}$$

$$-2\mathrm{Pr}_0 \frac{\partial}{\partial Y}\left[f_1 \frac{\partial V}{\partial Y}\right] - \mathrm{Pr}_0 \frac{\partial}{\partial X}\left[\left(f_1 \frac{\partial U}{\partial Y} + f_1 \frac{\partial V}{\partial X}\right)\right] - \lambda \frac{\partial}{\partial Y}\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) + \frac{\partial V}{\partial t^*} + \left[U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y}\right] - \mathrm{Ra}_0\, \mathrm{Pr}_0\, \theta = 0 , \tag{6}$$

$$-\frac{\partial}{\partial X}\left[f_2 \frac{\partial \theta}{\partial X}\right] - \frac{\partial}{\partial Y}\left[f_2 \frac{\partial \theta}{\partial Y}\right] + \frac{\partial \theta}{\partial t^*} + \left[U \frac{\partial \theta}{\partial X} + V \frac{\partial \theta}{\partial Y}\right] = 0 . \tag{7}$$

Here, X and Y denote the dimensionless horizontal and vertical co-ordinates, respectively; U, V, T are the dimensionless horizontal and vertical velocity components and temperature, respectively. Pr indicates the Prandtl number and Ra the Rayleigh number and the subscript 0 denotes parameters evaluated at a reference temperature state. The energy equation remains unchanged.

3 Solution procedure

The solution methods used depend on the model, computational resources, the non-linearity of the system, and the strength of the coupling between equations. The strong coupling between equations (e.g., momentum and energy) that is characteristic of convective heat transfer problems makes these combined equation methods optimal from the standpoint of convergence rate. The disadvantage, of course, is that a very large and computationally expensive

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matrix problem must be treated at each iteration. The requirement to perform larger (more elements and higher dimensionality) and more complex (physical phenomena) simulations has reached the point where usual direct matrix methods for combined equations are prohibitively expensive. A natural choice to make the matrix problem more affordable (while retaining the standard fixed point schemes) is to switch from the direct, Gauss elimination method to the iterative matrix methods, such as the pre-conditioned conjugate gradient (PCG) method. Unfortunately, the development of iterative methods for combined equation sets has been severely handicapped by the lack of good pre-conditioner techniques that can adequately treat the dominating effect of the incompressibility constraint. To make progress with current iterative methods, the combined equation approach must be sacrificed for alternative formulations of the discrete equations.

The presence of the convective terms in the momentum and energy equations makes the coefficient matrix asymmetric and non-linear. The convective terms depend on the velocity components and temperature, which are unknown. Consequently, an iterative technique must be used to solve the equations and the following procedure was employed in the present work.

At the beginning of the first iteration, the flow field is set to zero, and the coefficient matrices are evaluated and assembled. Then, the first iteration of the solution is obtained after imposing the specified boundary conditions of the problem. The calculated velocity and temperature fields are then used for the same Rayleigh number in the second iteration to re-evaluate the unknowns. The outlined procedure is repeated until the flow field computed in two consecutive iterations differs by less than the tolerance, *Tol*:

$$\sqrt{\sum_{i=1}^{N}\left|\Delta_i^{n+1}-\Delta_i^{n}\right|^2 \Big/ \sum_{i=1}^{N}\left|\Delta_i^{n+1}\right|^2} < Tol \tag{8}$$

where Δ_i^n denotes the flow variable at node i and iteration n, and N is the number of unknowns. A value of $Tol = 0.01$ was used in all cases. It is necessary to start with a low value of Rayleigh number (10^3 is satisfactory): once convergence has been obtained for the given value of Rayleigh number, its value is incremented by a factor of 10. The iterative procedure is repeated with the initial guess for the velocity and temperature fields being the converged solution for the previous value of Rayleigh number. To accelerate the convergence, a weighted sum of the last and current solutions is used:

$$\tilde{\Delta} = \delta\Delta^n + (1-\delta)\Delta^{n+1} \tag{9}$$

where δ is an acceleration parameter, $0.0 \le \delta < 1.0$. A value of $\delta = 2/3$ has been shown to give a satisfactory convergence rate [9]. The integrals are evaluated numerically with the aid of Gaussian quadrature. The resulting set of nonlinear equations is linearised using the Newton iteration technique. In the post-

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processing, the distribution of the stream function ψ can be derived using the velocity field by solving the Poisson eqn (10) separately:

$$\frac{\partial^2 \psi}{\partial X^2} + \frac{\partial^2 \psi}{\partial Y^2} = \frac{\partial U}{\partial Y} - \frac{\partial V}{\partial X} \tag{10}$$

subject to the appropriate boundary conditions. The local Nusselt number on the cylinder wall is evaluated from

$$\mathrm{Nu}_k = -\left(R_k \frac{\partial \theta}{\partial r} \right)_{r=R_k} \qquad k = \text{inner, outer} \tag{11}$$

and the mean Nusselt number is determined as

$$\overline{\mathrm{Nu}}_k = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Nu}_k \, d\phi \qquad k = \text{inner, outer} \tag{12}$$

4 Results and discussions

Comparisons are restricted to the case of concentric pipes with artificial conditions because variable properties have negligible effect on the primitive variables for air and water. A schematic diagram of the physical model, boundary conditions and co-ordinate system is shown in Figure 1.

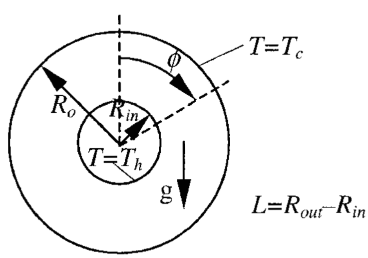

Figure 1. Diagram of the computational domain for the concentric pipes

Figures 2 and 3 illustrate isotherms and streamlines for a range of Rayleigh numbers. The velocity components U and V at $\phi = 90°$ are shown in Figures 4(a) and 4(b). As shown in Figure 4(b), for the case of constant properties the magnitude of the vertical component velocity V is higher near the hot surface and lower near the cold surface than for the temperature-dependent viscosity and temperature-dependent thermal conductivity cases.

Temperature distributions across the annulus at $\phi = 90°$, shows in Figure 4(c), are affected by both the thermal conductivity and the viscosity. However, the

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thermal conductivity effect is greater than the viscosity effect, and is in the opposite sense.

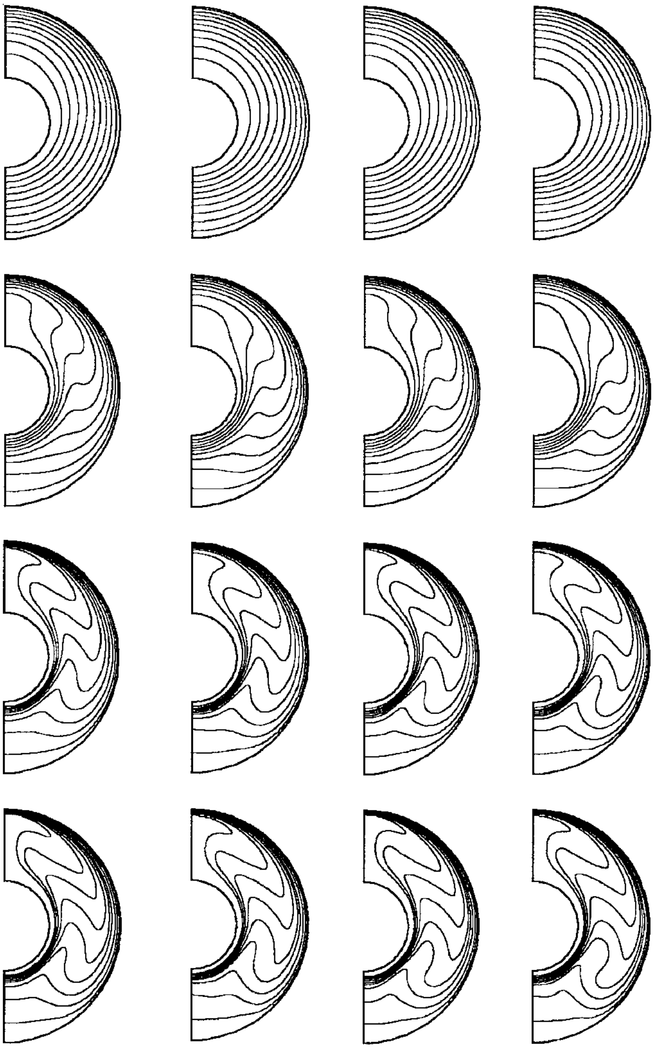
The local Nusselt number on each pipe, shown in Figure 4(d), is affected very significantly by the varying thermal conductivity but insignificantly by the viscosity.

$Ra_0 = 10^3$

$Ra_0 = 10^4$

$Ra_0 = 5\times10^4$

$Ra_0 = 10^5$



(a) Constant properties (b) $k\,(T)$ (c) $\mu\,(T)$ (d) $k\,(T)$ & $\mu\,(T)$

Figure 2. Isotherms for two-dimensional concentric pipes at various Rayleigh numbers and $Pr_0 = 1$, for constant and variable properties (artificial conditions).

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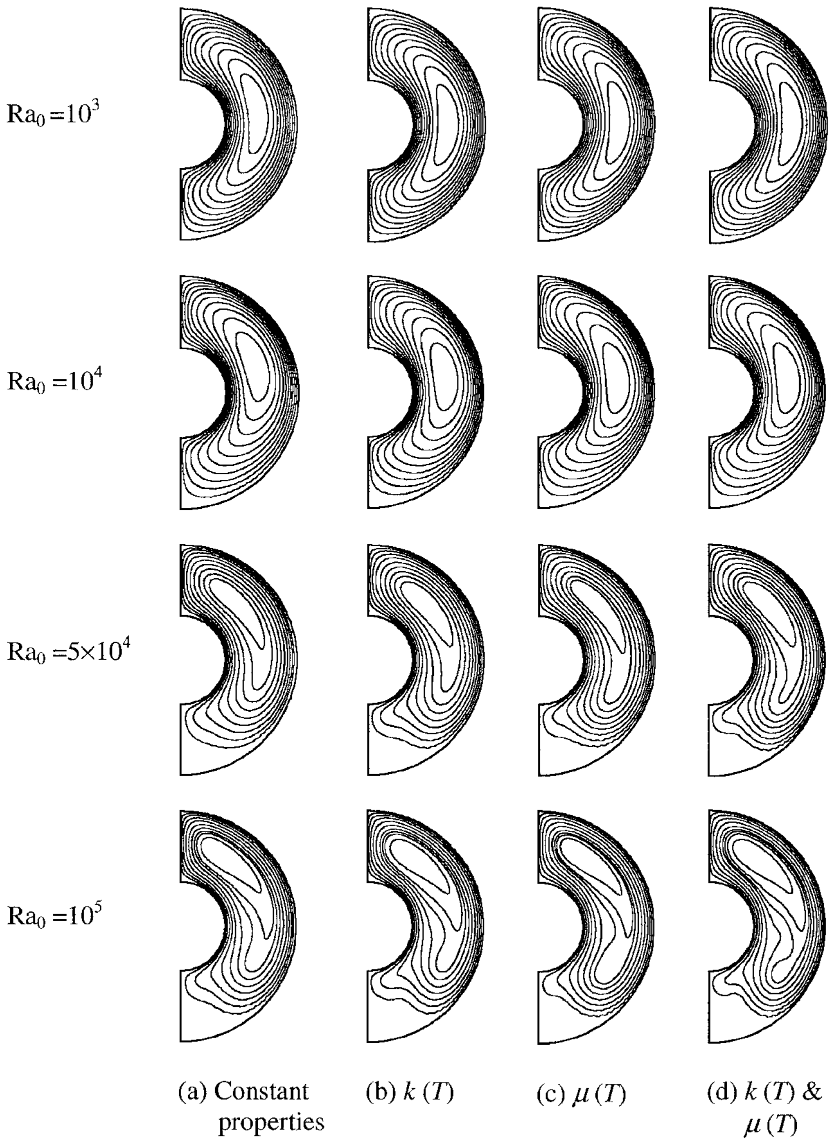

Figure 3. Streamlines for two-dimensional concentric pipes at various Rayleigh numbers and $Pr_0 = 1$, for constant and variable properties (artificial conditions).

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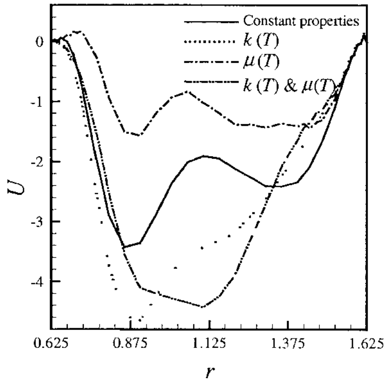
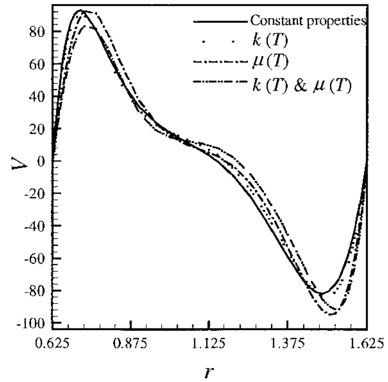
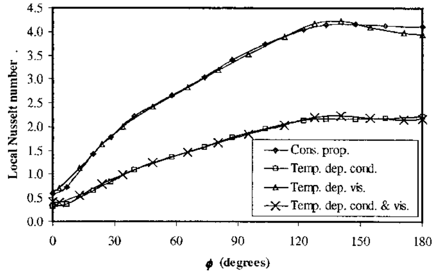
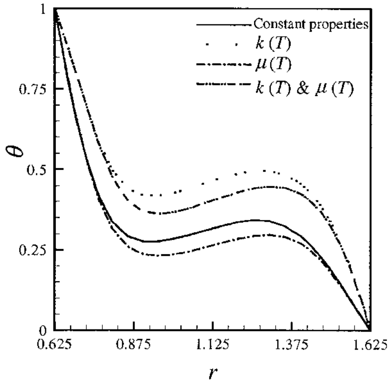


(a) Velocity U at $\phi = 90°$



(b) Velocity V at $\phi = 90°$





(c) Temperature θ at $\phi = 90°$

(d) Nusselt numbers for inner pipes

Figure 4. Distribution of U, V, θ at $\phi = 90°$ and local Nusselt numbers in inner pipe for concentric pipes at $Ra_0 = 5\times10^4$ and $Pr_0=1$, for constant and variable properties.

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5 Conclusions

A wide range of numerical results has been obtained to study the effect of property variation on natural convection heat transfer. When the thermal conductivity and viscosity are temperature-dependent, both the velocity components and temperature are affected by the property variation. However, the varying viscosity has most effect on the fluid velocity, while the effects of varying thermal conductivity were most noticeable in the temperature profiles and local Nusselt numbers. The effects of viscosity and thermal conductivity on temperature profiles are in the same direction for water, in contrast to gases where they are in the opposite direction. The effect of a simultaneous variation of viscosity and thermal conductivity can be estimated by combining their separate effects.

The results show that, over the temperature ranges considered, the temperature dependent viscosity and thermal conductivity of air and water have negligible effects on the natural convection. In terms of their effects on the overall Nusselt number over the temperature ranges employed, temperature-dependent viscosity causes a reduction in Nu of almost 0.5% compared with constant properties, variable thermal conductivity an increase of 0.2% and their combination a reduction of about 0.2%.

Some viscous liquids, such as glycerine and silicone oil, exhibit significant changes of viscosity and thermal conductivity with changing temperature. The simulation done assuming artificially large variation of properties has demonstrated that the program is capable of giving stable solutions in such cases.

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