Numerical methods for the determination of thermal properties by means of transient measurements

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Abstract

The determination of the thermal transport properties is realized by indirect measurements and a subsequent data analysis by solving an inverse problem. Measurements were executed by the Transient Hot Strip (THS) or the Transient Hot Wire (TWS) method. The measured transient voltage drop of an embedded thin metal foil or wire is related to the thermal conductivity and diffusivity by the time-dependent heat conduction equation. The resulting parameter identification was solved in two ways. On the one hand, in idealized case, an analytic approximation of the solution heat conduction equation is used for a linear procedure. When the conditions of the simplified model are checked very carefully, these solutions found are suitable for a parameter fit. Especially a reliable estimation of the allowable measurement time is given based on finite-element (FEM) simulations. On the other hand, for more general cases, a new nonlinear identification algorithm based on the FEM-solution of the heat conduction equation is introduced, which allows the approximation of the entire transient signal. The underlying numerical solution of the forward problem gives a precise description of reality, also in those situations, where an analytic approximative solution has not to be derived, so far, as in the case of layered materials. The results are applied to measured data.

1 Introduction

For practical purposes, for example in the building industry, the knowledge of the thermal properties of materials is of great interest. As it is infeasible to directly
measure these quantities, they have to be determined by the measurement of related quantities and solving an inverse problem.

Compared with steady state techniques, transient methods to determine the thermal conductivity and thermal diffusivity of solids offer substantial advantages like e.g., short measurement times and extended measurement and working temperature ranges [1]. In addition, the Transient Hot Wire (THW) and the Transient Hot Strip (THS) techniques allow a relatively simple set-up: A current carrying thin wire or strip is clamped between the two sample halves where it simultaneously acts as a resistive heater and a temperature sensor. From its temperature rise, $T(t)$, the above mentioned quantities are derived.

However, the working equations of transient techniques as described in Section 2 can be derived in closed form only for an idealized mathematical model of a heater of vanishing heat capacity embedded inside a sample of infinite extension. Due to both these major assumptions, the predicted signals deviate from the observed ones at short and long times. To find essential corrections to the closely related ideal models of the THW and THS techniques, numerical methods like finite-element method (FEM) calculations were performed (Section 3). For more general situations a new nonlinear algorithm for the inverse problem introduced in Section 4 is based on FEM forward calculations and an optimisation strategy. Methods of such kind were used for identification problems in other fields, too [2], [3]. This algorithm is as well applicable in those cases, where analytic or analytic approximative solutions do not exist or have not been derived, so far, as in case of layered materials. The results are in good agreement with experimental data obtained on the standard reference Pyrex [4].

2 A linear procedure for analysing transient signals

In the ideal THW model a line source completely gives off its Joule heat to the surrounding unbounded dielectric as the specific heat flow $U_0 I / L = \text{const.}$ The temperature rise $T(r,t)$ at a distance $r$ from the source at time $t$ is a measure of the sample’s thermal conductivity $\lambda$ and thermal diffusivity $\alpha$. In practice, the signal $T(r,t)$ is monitored as the voltage drop [1]

$$\Delta U(T(t)) = U(T(t)) - U_0 = \alpha U_0 (T(t) - T_0) = \frac{\alpha U_0^2 I}{2\sqrt{\pi}L\lambda} \cdot f(r).$$  \hspace{1cm} (1)

of a thin current carrying wire of length $L$ and radius $r$. $\alpha$ denotes the temperature coefficient of the electrical resistance of the wire, $U_0$ its offset voltage at time zero, and $\Delta T(t)$ the temperature increase. Eq. (1) results from an analytical approximation of the solution of the heat conduction equation. Here

$$\tau = \tau_c = \frac{\sqrt{4\pi at}}{r}$$  \hspace{1cm} (2)

and
For \( \tau_i^2 \ll 1 \), \((t_{\text{min}} = r^2/4a)\), the nonlinear and implicit function \((3)\) can be linearized with respect to \(\ln t\):

\[
\ln t + \ln \frac{4a}{r^2 C} = \exp \gamma \quad (\text{Euler's constant})
\]

Substituting Eq. (4) into Eq. (1) yields the fundamental working equation of the THW method. \(\lambda\) is derived from the slope and \(a\) from slope and intercept of the line \(\Delta U \text{ vs } \ln t\).

To derive the working equation of the THS method where a metal strip of width \(D\) is used in place of the wire, Eqs. (2) and (3) have to be replaced by

\[
f = f_s(\tau_D) = \tau_D \text{erf} \left( \frac{\tau_D}{2\sqrt{2}} \right) - \frac{\tau_D^2}{2\sqrt{2}} \left[ 1 - \exp \left( -\frac{\tau_D^2}{2} \right) \right] - \frac{1}{2\sqrt{\pi}} \text{Ei} \left( -\frac{\tau_D^2}{2} \right)
\]

and

\[
\tau = \tau_D = \frac{4at}{D},
\]

respectively [5]. For \(\tau_D^2 \leq 1/4\), \((t_{\text{min}} = D^2a)\), the shape function \(f_s(\tau_D)\) can be approximated as

\[
f_s(t) \approx \frac{1}{2\sqrt{\pi}} \left[ \ln t + \ln \frac{45a}{D^2 C} \right].
\]

Figure 1: A typical plot of a THS signal (strip-voltage) vs. \(\ln t\) measured on Pyrex with marked three characteristic regions.

Again substituting Eq. (7) into Eq. (1) results in the THS equation for an ideal model of an infinite sample [6]. In a real experimental situation, Eq. (1) is valid
only for such a time interval \([t_{\text{min}}, t_{\text{max}}]\) during which the heater's thermal influence can be confined to the (finite) sample itself. The individual value of \(t_{\text{max}}\) depends on the heat capacity of the heater and the dimensions and thermal diffusivity of the sample. It is not covered by theory but by numerical simulations as described in the next section. Fig. 1 shows the THS signal (strip-voltage) vs. In \(t\) measured on a Pyrex standard reference at a temperature of 80°C. Obviously, the plot can be divided into the three characteristic parts described above.

3 Numerical simulations of the transient signal

The temperature field inside the sample caused by the heated strip is given by the solution of the heat conduction equation

\[
\rho c_p \frac{\partial T(x,t)}{\partial t} = \lambda \Delta T(x,t) + q(x)
\]

\(T(x,0) = T_0\), \(x \in \Omega\)

for an isotropic homogeneous solid, where \(\rho\) is the density and \(c_p\) the specific heat assumed to be constant for small temperature differences. The heat source \(q\) [W/mm²] is limited to the volume of the strip. Experimental tests showed that the measurement configuration allows a two-dimensional treatment of the problem. For a sufficiently long strip (length \(l \geq 100\)mm) the boundary effects at both ends of the strip are neglable. Eq. (8) therefore is defined in a cross section perpendicular to the strip. This applies by analogy to a wire as a heat source.

Due to its nonvanishing heat capacity, the strip stores heat at the beginning of the experiment. In the course of the experiment still more energy dissipates at the outer boundary, i.e. at the exposed surface of the sample. The thermal complex behavior of strip and sample introduces initial and boundary conditions that cannot be treated by the ideal model described in Section 2. All effects may be considered by Eq. (8), but, in those cases, an analytic solution is not available.

We use a 2D FEM including an effective grid adaptation strategy. Because of the symmetric situation of the heat conduction caused by the inner source, the integration domain is reduced to a quarter of the sample. On the cut boundaries, homogenous Neumann conditions are assumed. To fill the area of the very thin strip (thickness 0.01mm) with complete triangles, we created a special layered grid. In the other case, a small area including the wire is refined four times compared to the boundary area. The transient signal is calculated by averaging the time-dependent temperature over the cross-section of the strip or wire.

The boundary conditions belong to (8) are assumed to be of the third kind

\[
-\lambda \left( \frac{\partial T(x,t)}{\partial n} \right) = k(T(x,t) - T_0)\), \(x \in \partial \Omega, 0 \leq t \leq b\).
\]

For \(k = 0\), the heat flux vanishes, or in other words, a total insulation is described, we have adiabatic conditions i.e. homogenous conditions of the second kind. For
Figure 2: The simulated THW signals (given as temperature rise) for six samples with different radii $R$ [mm], each under isothermal (lower curves) and under adiabatic conditions (upper curves). In the middle of the curves, the theoretical signal (TS) is included.

$h \to 0$, the heat flux has to remain finite, therefore $T(x,t) = 0$ must be kept on the boundary and Eq. (9) assumes the isothermal condition. In reality, $0 < h < \infty$ is valid, and the actual value depends on the experimental set-up. The different influence of the both extreme boundary conditions on the THW signal is shown in Fig. 2. The temperature differences for six cylindrical samples with the same thermal properties ($\alpha = 1 \text{ mm}^2/\text{s}, \lambda = 1 \text{ W/Km}$) but different size (radius $R = 3; 5; 10; 20; 30; 50$ mm) each under adiabatic and under isothermal boundary conditions, were simulated and the curve of the ideal infinite model (infinite sample and idealized source) is also added. The THW signals for any given parameter $h$ in the boundary condition (9) can be expected between the two special cases: adiabatic and isothermal. Obviously, the point where the two curves diverge indicates the time at which the influence of the sample surface begins. The same is true for the THS signal. More precisely, we define the maximum time $t_{\text{max}}$ for which the linear working equation is valid (see Sect. 2) by the equation

$$TOL = T_{\text{ad}}(t_{\text{max}}R) - T_{\text{iso}}(t_{\text{max}}R) = \varepsilon$$

where $T_{\text{ad}}(t,R)$ and $T_{\text{iso}}(t,R)$ denotes the signals for the adiabatic and the isothermal case, respectively, depending on the sample radius. However, for $t_{\text{max}}$ satisfying Eq. (10) and a realistic parameter $h$ in Eq. (9), the deviation of the real signal from the theoretical signal in an infinite sample will be smaller than $\varepsilon$. In analogy to the situation for point sources, a proportionality

$$t_{\text{max}} \sim R^2 / \alpha$$
may be expected. However, the factor of proportionality $f_{ac}$ can be determined by numerical experiments only. Tab 1 shows some of our THS results for $t_{\text{max}}$ and $f_{ac}$ in dependency of the sample radius. Here a realistic tolerance value $\varepsilon = 0.01$ and a thermal diffusivity $\alpha = 1 \text{ mm}^2/\text{s}$ were assumed, but $f_{ac}$ is independent of $\alpha$.

<table>
<thead>
<tr>
<th>$R$ [mm]</th>
<th>$t_{\text{max}}$ [s]</th>
<th>$f_{ac}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.2</td>
<td>0.133</td>
</tr>
<tr>
<td>5</td>
<td>4.2</td>
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<td>0.184</td>
</tr>
<tr>
<td>50</td>
<td>457.6</td>
<td>0.183</td>
</tr>
</tbody>
</table>

Tab 1. Results for the maximal allowable measurement time $t_{\text{max}}$ and the factor of proportionality $f_{ac}$ in relation (11), both depending on the sample radius $R$. For small radii (3 and 5mm), the relation (11) does not furnish accurate results because of the finite extension of the strip. In case of rectangular samples, $R$ has to be replaced by the minimum distance between the centre of the sample and its surface. The simulations for the THW technique yield the same factor $f_{ac}$. Together with $f_{ac} = 0.18$ and a coarse estimation of $\alpha$ (rounded up), we get

$$t_{\text{max}} = 0.18 \frac{R^2}{\alpha},$$

which is a good advance estimation of the maximal allowable measurement time.

Figure 3: Simulated THS signals (given as temperature rise) with (lower) and without (upper) considering the special strip properties.
Additionally to the definition of $t_{\text{max}}$, an other problem is given with the difference in the materials of the heat source and the sample, having different thermal properties, which is not considered in the ideal model. Fig 3 shows two simulated THS signals, for the lower curve for the strip the material properties of nickel ($a = 18.7 \text{ mm}^2/\text{s}$, $\lambda = 69.1 \text{ W/km}$) are considered while for the upper one they are not. The physical reason for the difference of the both curves lies in the different temperature distribution inside the strip as shown in Fig. 4 and therefore, the resulting different heat emission to the sample. Nevertheless, the difference is covered by the measurement uncertainty.

![Graph](image)

**Figure 4:** Temperature distribution cross a part of the sample including the strip with the centre at $x = 20 \text{ mm}$ and the end at $x = 18.75 \text{ mm}$.

### 4 An inverse procedure based on FEM – Simulations

#### 4.1 The Algorithm

The numerical simulation of the vector $T^{\text{sim}}$ of the discrete measurement signal is the solution of the so-called forward problem for a given time vector $t$. For given thermal properties, geometry of the heat source and the sample and given boundary conditions, first, the heat conduction equation (8) is solved, where subsequently the temperature profile of the cross section of the strip is averaged. In practice, the signal is the voltage drop. The relation between it and the temperature is quite simple (Eq. (1)), so in the algorithm we have to do with the temperature only. The measurement data set $T^{\text{meas}}$ is converted in temperature values.

In the inverse problem, the thermal properties and the data change their position, the data are known and the properties are to be determined. Eq. (8) with the boundary condition (9) and its numerical solution allow the vector $T^{\text{sim}}(\lambda, a)$ to be simulated under the assumption that $\lambda$ and $a$ are the actual thermal properties of the sample. The aim now is to find two values for $\lambda$ and $a$, which provide an
optimal fit between the simulated data $T^\text{sim}(\lambda, a)$ and the measured data $T^\text{mes}$. The essential difference to the linear procedure lies in the possibility to fit the whole signal curve instead of the linear part, only.

The basic strategy of the approximation consists in an iterative correction of the thermal parameters and can be demonstrated by the formal iteration procedure:

1. Choose an initial approximation $\lambda$ and $a$.
2. Solve the forward problem, i.e. compute $T^\text{sim}(\lambda, a)$.
3. Compare $T^\text{sim}(\lambda, a)$ with $T^\text{mes}$; if $\|T^\text{sim}(\lambda, a) - T^\text{mes}\| < \varepsilon_0$, then go to step 5.
4. Correct $\lambda$ and $a$; go to step 2.
5. End.

The iteration process starts with an initial guess for $\lambda$ and $a$. The norm to compare the vectors of simulated and measured data in the third step of the optimization problem is the $l_2$ norm:

$$\|T^\text{sim}(\lambda, a) - T^\text{mes}\|^2 = \sum_{i=1}^{n} \|T^\text{sim}_i(\lambda, a, t_i) - T^\text{mes}_i(t_i)\|^2 = \sum_{i=1}^{n} F_i^2$$

Here $n$ is the number of measurement data taken into account in the identification algorithm. The optimization problem reads now

$$\sum_{i=1}^{n} F_i = \text{min!}$$

which is solved by the Levenberg–Marquardt method from the program library of the International Mathematical Subroutine Library (IMSL). It combines the Gauss–Newton method with the gradient method:

$$p_{k+1} = p_k - \left( (F')^T F' \right) (p) + \kappa_k I^{-1} \left( (F')^T F \right) (p)$$

Here $p_k = (\lambda, a)$ is the parameter vector of the thermal properties of the $k$-th iteration, $I$ the unit matrix, and $\kappa_k$ a numerical control parameter. An iteration corresponds to a parameter correction that agrees with step 4 of the formal procedure. Because of its trust region approach this method is appropriate for handling ill-conditioned problems. In case of layered samples, $\lambda$ and $a$ are vectors, each component corresponds to a layer.

4.2 An Example

The THS measurement data for Pyrex (see Fig. 1) serve as a test series for the given algorithm as well as for a comparison with the linear procedure (Sec. 2). The sample has a cross-section of $40\times40\text{mm}^2$ and that of the nickel strip in the middle is $3.5\times0.01\text{mm}^2$. As mentioned above, for reasons of symmetry, we have to calculate the temperature distribution in a quarter only. The corresponding
FEM grid contains 5308 triangles and 2763 nodes, which allows a direct solver to be used.

The algorithm allows to take the entire temperature curve into account, but we have not done so for two reasons. First, due to experimental circumstances, the signal did not agree well enough with the theoretical prediction within a short time interval after the start of the experiment, which, afterwards, is omitted. The related effect on the result is slight, but can be excluded this way. Secondly, in general the parameter $h$ in the boundary condition (9) is precisely not know. It was included in the parameter set to be identified, but the condition of the problem was extremely bad. The incorporation of regularization techniques for this case is planed. A good compromise is the restriction to a time interval where the sample boundary has no influence on the transient signal. The algorithm was tested starting from different initial guesses but this did not influence the solution: $\lambda = 1.22 \text{ W/mK}$ and $a = 0.693 \text{ mm}^2/\text{s}$. The application of the linear procedure (Sec. 2) considering the time interval $[26.6, 107.18]$ for the linear fit according to (1) and (7) give the result: $\lambda = 1.23 \text{ W/mK}$ and $a = 0.668 \text{ mm}^2/\text{s}$. And as we see, the two results are in a good agreement. With the result for $a$, we determine the time interval corresponding to Section 2 and 3 at $[t_{\text{min}}, t_{\text{max}}] = [18.4, 107.8]$, which confirms the estimation. Fig. 5 shows the results comprising the measurement signal and two simulated curves, meeting isothermal and adiabatic boundary conditions, respectively and the identified thermal parameters.

Figure 5: Comparison between the measurement signal for Pyrex and simulated temperature curves as result of the parameter identification (isothermal and adiabatic boundary conditions (BC))
5 Conclusions

The Transient Hot Strip as well as the Transient Hot Wire methods are corresponding techniques for the simultaneous indirect measurement of the thermal transport properties. The linear procedure for analyzing the transient signals is based on some idealizations of the underlying physical model. On the other hand, a realistic model needs a more extensive numerical method for the solution of heat conduction equation, and the nonlinear identification procedure a more complex optimization strategy. With the second variant, the first one was verified and open problems were solved. For the linear procedure a time interval is required where the signal should be linear vs. \( t \), but its upper limit is not precisely defined. Now we have derived an analytic expression for its determination.

For homogenous materials the mathematical principles could be defined. In case of layered material as well as sandwich constructions the linear procedure reaches its limits, but the inverse algorithm offers a good prospect for the determination of the different thermal transport properties.

References