3D topology optimization using cellular automata

E. Kita\textsuperscript{1}, H. Saito\textsuperscript{2}, T. Kobayashi\textsuperscript{1} and Y. M. Xie\textsuperscript{3}

\textsuperscript{1}Graduate School of Information Sciences, Nagoya University, Nagoya 464-8301, Japan
\textsuperscript{2}Graduate School of Human Informatics, Nagoya University, Nagoya 464-8301, Japan
\textsuperscript{3}RMIT University, Australia

Abstract

This paper describes the three-dimensional structural design based on the concept of cellular automata. The object domain under consideration is divided into small cube elements to perform stress analysis by the finite element method. The density of each element is taken as the design variable. The density is updated with the local rule which is defined as the local relationship of the variable and the stress distribution. The present scheme is applied to the design of a three-dimensional structure.

1 Introduction

In the cellular automata simulation, the design domain is divided into small square cells at which the state variables are specified. The state variable is updated at each time step according to the local rule in order to simulate the phenomenon. The local rule is defined as the local relationship between the neighboring cells alone and therefore, the cellular automata simulation is especially effective for the simulation of the phenomenon of which the governing differential equation can not be defined.

Some researchers have been studying the application of the cellular automata simulation to the structural optimization [1, 2, 3, 4]. Authors have applied the cellular automata simulation or the local rule for design of the two-dimensional structures [5]. In this scheme, the use of the special constraint condition transforms the global optimization problem for the whole structure into the local optimization...
problem for each cell. The penalty function is defined from two objective functions and the constraint condition. The local rule is derived from stationalizing the penalty function.

In this paper, the optimization scheme for the two-dimensional structures is extended to the design of the three-dimensional structures. The density distribution of the structure is taken as the design variable. One takes as the global objective functions for the whole structure, the minimization of the whole structure and the reduction of the equivalent stress below the reference stress. The special constraint transforms the global optimization problem into the local optimization problem. Finally, the scheme is applied to the design of the simple three-dimensional structures.

2 Finite element method

2.1 Basic relationships

We explain the finite element formulation briefly so that the derivation of the local rule in the next section may easily be understood [6, 7].

The principle of the virtual work without the body forces is given as

$$\int_{\Omega} \delta \varepsilon^T \sigma d\Omega = \int_{\Gamma_t} \delta u^T \bar{t} d\Gamma$$

where $\Omega$, $\Gamma_u$ and $\Gamma_t$ respectively denote the domain occupied by the object under consideration, its displacement- and traction-specified boundaries and the whole boundary $\Gamma = \Gamma_u \bigcap \Gamma_t$. The vectors $u$, $t$, $\varepsilon$ and $\sigma$ respectively denote the vectors of the displacement, traction, strain and stress components in the two-dimensional elastic problem. The symbol $\delta$ and the superscript $T$ respectively denote the virtual changes and the transpose of the matrix or vector.

The relationship between the displacement and the strain components are given as

$$\varepsilon = Au$$

where $A$ denotes the matrix of the differential operators.

The relationship between the strain and the stress components is given as

$$\sigma = E D \varepsilon$$

where $E$ denotes Young’s modulus and $E D$ denotes the stiffness matrix.

The relationship between the stress and the traction components are as

$$t_i = \sigma_{ij} n_j$$

where $n_j$ denotes the $x_j$-component of the outer normal vector on the boundary.
2.2 Discretization

We shall consider the discretization of equation (1).

The displacement components at element $e$ are approximated by the interpolation functions $N$ with the nodal displacements $U_e$;

$$u_e = NU_e$$ (5)

The stress and the strain components may then be expressed as

$$\varepsilon_e = Au_e = ANU_e = BU_e$$ (6)

$$\sigma_e = E\varepsilon_e = EDBU_e$$ (7)

Discretizing equation (1) and substituting the above equations, finally we have

$$\sum_{e=1}^{N_e} \delta U_e^T h_e E K_e U_e = \sum_{l=1}^{N_l} \delta U_l^T f_l'$$ (8)

and

$$\delta U^T K U = \delta U^T f$$

$$K U = f$$ (9)

where $K$ and $f$ respectively denote the global stiffness matrix and the global equivalent nodal force vector.

3 Derivation of local rule

3.1 Cellular representation of three-dimensional structure

The three-dimensional structure is uniformly divided into cube elements. While the cube elements are used for the stress analysis by the help of the finite element analysis, they act as “cells” in the cellular automata simulation. All cells neighboring to the cell are referred as the neighborhood cells for the cell.

The cell relationship should be defined in order to define the local rule. We assume as “cell$_0$”, the cell of which the state variable is updated according to the rule and 26 cells on the Cell$_0$ are taken as the “neighborhood cells”, which are numbered as the “cell$_1$, cell$_2$, ⋮, cell$_{26}$”. When the cell$_0$ is on the boundary, there are less than 26 neighborhood cells and therefore, one defines the existing cells alone as the neighborhood cells.
3.2 Optimization problem

The density of each cell is taken as the design variable. As the objective functions, one considers the minimization of the whole structure and the reduction of the equivalent stress below the reference stress. The objective functions are defined as

\[ W_1 = \frac{1}{2} \left( \frac{\hat{\rho}}{\hat{\rho}_0} \right)^2 \equiv \frac{1}{2} \rho^2 \]  
\[ W_2 = \frac{1}{2} \left( \frac{\hat{\sigma}}{\sigma} - 1 \right)^2 \equiv \frac{1}{2} (\sigma - 1)^2 \]  

where \( \hat{\rho}, \hat{\rho}_0, \hat{\sigma} \) and \( \sigma_0 \) denote the density of the cell \( 0 \), its initial value, the stress at the cell \( 0 \) and its initial value, respectively.

We shall introduce the following constraint condition in order to define the local rule.

\[ g_j = \frac{\hat{\sigma}_j}{\sigma^0_j} - 1 \equiv \sigma_j - 1 \equiv 0 \quad (j = 1, \ldots, N) \]  

where \( \hat{\sigma}_j \) and \( \sigma^0_j \) denote the equivalent stresses at the cell \( j \) at the time step \( k \) and \( k - 1 \), respectively. \( N \) denotes the total number of the neighborhood cells on the cell \( 0 \).

The penalty function of the optimization problem is defined from equations (10), (11) and (12) as follows.

\[ W = \alpha W_1 + \beta W_2 + \frac{p}{2} \sum_{j=1}^{N} g_j^2 \]
\[ = \frac{\alpha}{2} \rho^2 + \frac{\beta}{2} (\sigma - 1)^2 + \frac{p}{2} \sum_{j=1}^{N} (\sigma_j - 1)^2 \]  

where \( p \) is the penalty parameter. Besides, \( \alpha \) and \( \beta \) are the weight parameters, which are defined as

\[ \alpha + \beta = 1 \]
\[ \beta = \{ \sigma (\sigma < 1), 1 (\sigma \geq 1) \} \]  

Equation (13) is expanded around \( \rho + \delta \rho \) and stationized as follows.

\[ \delta \rho = -\frac{\alpha \rho + \beta (\sigma - 1) \dot{\sigma} + p \sum (\sigma_j - 1) \dot{\sigma}_j}{\alpha + \beta \dot{\sigma}^2 + p \sum \dot{\sigma}_j^2} \]
3.3 Approximate design sensitivity and local rule

The introduction of the special constraint condition simplifies the estimation of the design sensitivity. The following relationship between the material density $\rho$ and the Young’s modulus $E$ is held;

$$E = c_1 \rho^{c_2}$$  \hspace{1cm} (16)

where $c_1$ and $c_2$ means the constants; $c_1 = 1$, $c_2 = 2$.

Direct differentiation of the above equation with respect to the density $\rho$ leads to

$$\frac{\partial E}{\partial \rho} = c_1 c_2 \rho^{c_2-1} = \frac{c_2 E}{\rho}$$ \hspace{1cm} (17)

Differentiating equation (7) with respect to $\rho$ and substituting the above expression, we have

$$\dot{\sigma} = \frac{\sigma}{\rho} = \frac{\partial E}{\partial \rho} D\varepsilon = \frac{c_2 E}{\rho} D\varepsilon = \frac{c_2 \sigma}{\rho}$$ \hspace{1cm} (18)

We shall assume that the stress $\sigma_j$ at the cell $j$ is in inverse proportion to the density of the cell 0. So, we have

$$\frac{\partial \sigma_j}{\partial \rho} = -\frac{\sigma_j}{\rho}$$ \hspace{1cm} (19)

Substituting equations (18) and (19) into equation (15) leads to

$$\delta \rho = -\frac{\alpha \rho^2 + \beta (\sigma - 1) \sigma - p \sum (\sigma_j - 1) \sigma_j}{\alpha \rho^2 + \beta \sigma^2 + p \sum \sigma_j^2} \rho$$ \hspace{1cm} (20)

The density $\rho$ is updated by equation (20) as follows.

$$\rho^{k+1} = \rho^k + \delta \rho$$ \hspace{1cm} (21)

where the superscript $k$ denotes the time step.

3.4 Optimization algorithm

The optimization algorithm of the present scheme is as follows.

1. Input initial data.
2. Perform the stress analysis with the finite element method.
3. Check the convergence condition. If the condition is satisfied, output the results. If not satisfied, go to the next step.
4. Estimate \( \delta \rho \) in order to the design variable.
5. Go to step 2.

The finite element analysis is performed with the open source software FElt version 3.0 [8]. Besides, the convergence condition is not specified in the numerical examples.

![Figure 1: Object under consideration.](image)

Table 1: Design conditions. \( \sigma_{0}^{max} \) means the maximum stress at the initial analysis.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus</td>
<td>( E = 1.0 \times 10^{10} ) (Pa)</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>( \nu = 0.3 )</td>
</tr>
<tr>
<td>External load</td>
<td>( P = 1 ) (N)</td>
</tr>
<tr>
<td>Initial density</td>
<td>( \rho_0 = 1 ) kg/m(^3)</td>
</tr>
<tr>
<td>Penalty parameter</td>
<td>( p = 1 )</td>
</tr>
<tr>
<td>Reference stress</td>
<td>( \sigma_c = 0.8 \times \sigma_{0}^{max} )</td>
</tr>
</tbody>
</table>

4 Numerical example

The initial assumed profile of the structure is shown in Fig.1. The plane ABCD is fixed in all directions and the external load \( P \) is applied on the center of the plane EFGH in the \( y \)-direction. The object domain is uniformly divided into 200 cube elements. The initial conditions are shown in Table 1. The same initial density is specified at all cells.

The convergence histories of the stress and the total weight of the structure are shown in Fig.2. The left- and right-ordinates denote the stress divided with the reference stress and the total weight of the structure divided with the initial
Figure 2: Convergence histories of stress and weight.

Figure 3: Final profile (Side view).

weight, respectively. The abscissa the number of iterations. The total weight and the maximum stress finally converge to 10% of the initial weight and the reference stress, respectively. Therefore, we can say that the final structure almost satisfy the design requirements.
The side view, the top view, the view from the plane ABCD and the view from the plane EFGH are shown in Figs. 3, 4, 5 and 6, respectively. The final profile is of thinner width than the initial one and of relatively simple profile connecting the fixed and the load points.
5 Conclusions

This paper described the design of the three-dimensional structures using local rule. The design domain is divided into many small cubes (cells) to perform the finite element analysis. The density of each cube (cell) is taken as the design variable. The cell density is updated according to the local rule from the stress states at the cell and its neighborhood cells. The iterative application of the local rule finally generates the three-dimensional structure satisfying the design requirements. Finally, the present method was applied to some numerical examples in order to discuss its property.

References


