Nonlinear finite element modelling of composite structures with integrated piezoelectric layers

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Abstract

A geometrically nonlinear composite shell element with integrated piezoelectric layers is presented. Three translational and two rotational nodal degrees of freedom are used for the first-order transverse shear approximation. The finite element is tested on a number of static problems using the piezoelectric layers as actuators as well as sensors. The numerical approximations to these static problems are obtained using the total Lagrangian formulation for the moderate rotation shell theory. The significance of a geometrically nonlinear formulation in order to predict the sensor properties is shown.

Keywords: geometrical nonlinearity, piezoelectricity, actuators, sensors.

1 Introduction

In recent years much research has been conducted in theoretically, numerically and experimentally investigating the possibilities that integration of piezoelectric material into smart structures can bear. Geometrically linear theories and numerical methods have been developed by many authors. For instance Crawley and de Luis [1] or Robbins and Reddy [2] studied elastic smart beams. Piefort [3] has pointed out that the application of the beam theory to model smart structures is not satisfying when collocated systems are considered. Tzou and Tseng [4], Batra et al. [5] and Lammering [6] among many other researchers developed methods to investigate piezointegrated plate and shell structures.

Considerably less work can be found in the area of geometrically nonlinear modelling. This is quite surprising considering the fact that many authors refer to a benchmark problem in which a 1mm thin cantilever bimorph beam is subjected to a 1cm transverse tip deflection. Nevertheless some literature can be
found, e.g. Mukherjee and Chaudhuri [7], Yi et al. [8], Shi and Atluri [9] and Chróscielewski et al. [10, 11].

In this work a moderate rotation theory of composite shells with integrated piezoelectric layers is developed. Special attention is given to the proper definition of electric field vectors for a total Lagrangian approach to geometrically non-linear structural problems. The described method is deployed at two benchmark problems including a piezoelectric bimorph beam and three types of C-block actuators. It is demonstrated that the effect of the geometrical nonlinearity is far from negligible especially when the piezoelectric layers are used as sensors.

2 Composite shell theory with piezoelectric layers

2.1 Basic assumptions and nomenclature

The implementation of nonlinear theories necessitates the application of incremental finite element formulations. In this paper the total Lagrangian formulation is applied which focusses on the following three body configurations $m\mathcal{C}$ [12]: the initial configuration $0\mathcal{C}$, the actual configuration $1\mathcal{C}$ and the searched configuration $2\mathcal{C}$. Quantities which belong to a particular configuration are denoted with a left superscript.

The coordinates $\Theta^1$ and $\Theta^2$ are introduced as a parametric representation of the shell undeformed midsurface $0\Omega$ and $\Theta^3$ is the thickness coordinate. According to the first-order shear deformation or Reissner-Mindlin theory it is assumed that straight lines perpendicular to the undeformed midsurface remain straight, inextensible but not perpendicular during deformation. Therefore the displacement vector...
\( m \tilde{V} \) in an arbitrary configuration \( m \mathcal{C} \) can be expressed by

\[
m \tilde{V} = m \tilde{V}^0 + \Theta^3 m \tilde{V}^1
\]

with

\[
m \tilde{V}^0 = m \tilde{V}^0_a g_i = m \tilde{V}^0 a_i \quad \text{and} \quad m \tilde{V}^1 = m \tilde{V}^1_\alpha g_\alpha = m \tilde{V}^1 a^\alpha
\]

The co- and contravariant base vectors of the midsurface are denoted by \( g_i \) and \( a^i \), respectively. Einsteinian summation is applied for equal indices where Latin indices are defined as \( i \in [1, 2, 3] \) and Greek indices as \( \alpha \in [1, 2] \). Assuming the following orders of magnitude \[13\]:

- small strains \( \varepsilon_{ij} = \mathcal{O}(\theta^2) \) with \( \theta^2 \ll 1 \)
- small rotations about the normal \( \omega_{\alpha \beta} = \mathcal{O}(\theta^2) \)
- moderate rotations of the normal \( \omega_{\alpha 3} = \mathcal{O}(\theta) \),

the general Green-Lagrange strain displacement relationship

\[
\varepsilon_{ij} = \frac{1}{2} \left( V_{ij} + V_{ji} + V_{k|i} V_{k|j} \right),
\]

with \( V = V^i g_i = V_i g^i \) (\( g_i \) and \( g^i \) are the co- resp. contravariant base vectors in space), can be approximated by the following set of equations \[13\]:

\[
\begin{align*}
\varepsilon_{\alpha \beta} &= \varepsilon_{\alpha \beta}^0 + \Theta^3 \varepsilon_{\alpha \beta}^1 + (\Theta^3)^2 \varepsilon_{\alpha \beta}^2 \\
\varepsilon_{\alpha 3} &= \varepsilon_{\alpha 3}^0 + \Theta^3 \varepsilon_{\alpha 3}^1 \\
\varepsilon_{33} &= 0
\end{align*}
\]

with

\[
\begin{align*}
\varepsilon_{\alpha \beta} &= \theta_{\alpha \beta} + \frac{1}{2} \varphi_\alpha \varphi_\beta, \\
\varepsilon_{\alpha \beta}^1 &= \frac{1}{2} \left( v_\alpha |_{\beta} + v_\beta |_{\alpha} - b_\alpha^\lambda \varphi_\lambda \beta - b_\beta^\lambda \varphi_\lambda \alpha + \varphi_\alpha b_\lambda^\beta v_\lambda + \varphi_\beta b_\lambda^\alpha v_\lambda \right), \\
\varepsilon_{\alpha \beta}^2 &= \frac{1}{2} \left( b_\alpha^\lambda b_\beta^\kappa v_\lambda v_\kappa \right), \\
\varepsilon_{\alpha 3} &= \frac{1}{2} \left( 0 \varphi_\alpha + v_\alpha + b_\lambda^\alpha \varphi_\lambda \right) \quad \text{and} \quad \varepsilon_{33} = \frac{1}{2} v_\lambda v_\lambda |_{\alpha}.
\end{align*}
\]

The following abbreviations were used:

\[
\begin{align*}
\theta_{\alpha \beta} &= \frac{1}{2} \left( v_\alpha |_{\beta} + v_\beta |_{\alpha} \right) - b_{\alpha \beta} v_3, \\
\varphi_{\alpha \beta} &= v_{\alpha |\beta} - b_{\alpha \beta} v_3 \quad \text{and} \quad \varphi_\alpha = v_{3, \alpha} + b_\alpha^\lambda v_\lambda,
\end{align*}
\]

where \( b_{\alpha \beta} \) and \( b_\beta^\alpha \) denote the covariant and mixed components of the curvature tensor and \( (\cdot) |_{\alpha} \) is the covariant derivative with respect to the surface parameter \( \Theta^\alpha \).
2.2 Total Lagrangian formulation

In this work a decoupled electromechanical system, in which the electrical equilibrium is satisfied identically, is considered. Assuming that the configuration $2C$ represents a state of equilibrium, the virtual work principle states

$$2\delta W_i - 2\delta W_e = 0,$$

(4)

where $2\delta W_i$ and $2\delta W_e$ are the internal and external virtual work, respectively. In a total Lagrangian approach, i.e. when all quantities in the configuration $2C$ are referred to the initial configuration $0C$,

$$2\delta W_i = \int_{0V} 20s^{kl}\delta 20\varepsilon_{kl}0dV;$$

(5)

$$2\delta W_e = \int_{0V} 0\rho 20\delta V_0^0dV + \int_{0S} 20s^i\delta V_0^0dS,$$

(6)

where $20s^{kl}$ are the components of the second Piola-Kirchhoff stress tensor, $20\varepsilon_{kl}$ are the components of the Green-Lagrange strain tensor, $0\rho$ and $20B^i$ denote the mass density and body force vector components and $20s^i$ stands for the external prescribed stress vector components.

The application of the total Lagrangian formulation requires the introduction of field variables which refer to the undeformed configuration. In a similar fashion the electric quantities, which occur in the piezoelectric constitutive equation (7)-(8), have to be referred to the initial configuration. Therefore the electric displacement and the electric field vector are written as

$$0\sim D = J\overline{F}^{-1}t\sim D$$

and

$$0\sim E = -\text{GRAD}(\phi),$$

where $\overline{F}$ is the deformation gradient, $J$ its determinant, GRAD the gradient in the undeformed configuration and $\phi$ the electric potential. Considering the internal electric virtual work $\delta W_{ei}$ it can be shown that both electric vectors are energetically conjugated:

$$\delta W_{ei} = \int_{tV} t\sim D^Tt\sim \delta E^t0dV = \int_{0V} 0D^T0\delta E_0^0dV.$$

In the present approach we will assume that the piezoelectric layers are thin, i.e. we will assume that for actuators $E = \text{const.}$, and for sensors $E = 0$. This, in fact, leads to a decoupled analysis and $\delta W_{ei}$ does not appear in equation (5).

2.3 Constitutive relations

The piezoelectric effect will be described by the following set of linear constitutive equations in vector form:

$$\{0D\} = [e] \{0\varepsilon\} + [\delta] \{0E\}$$

(7)
\[ \{0S\} = [e] \{0\epsilon\} - [e]^T \{0E\} \tag{8} \]

where \(\{0S\}\) denotes the stress vector, \(\{0\epsilon\}\) the strain vector, \(\{0D\}\) the electric displacement vector and \(\{0E\}\) the electric field vector.

\[
\{0S\} = \begin{bmatrix}
\sigma^{11} \\
\sigma^{22} \\
\tau^{12} \\
\tau^{23} \\
\tau^{13}
\end{bmatrix},
\{0\epsilon\} = \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
2\varepsilon_{12} \\
2\varepsilon_{23} \\
2\varepsilon_{13}
\end{bmatrix},
\{0D\} = \begin{bmatrix}
D^1 \\
D^2 \\
D^3
\end{bmatrix},
\{0E\} = \begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix}
\]

Further \([e] = [d][c]\) and \([e]^T = [c][d]^T\), \([c]\) denotes the elasticity matrix for anisotropic material, \([d]\) the piezoelectric constant matrix and \([\delta]\) the dielectric constant matrix.

\[
[c] = \begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 \\
c_{12} & c_{22} & c_{23} & 0 & 0 \\
c_{13} & c_{23} & c_{33} & 0 & 0 \\
0 & 0 & 0 & c_{44} & c_{45} \\
0 & 0 & 0 & c_{45} & c_{55}
\end{bmatrix},
[d]^T = \begin{bmatrix}
0 & 0 & d_{31} \\
0 & 0 & d_{31} \\
0 & 0 & 0 \\
0 & d_{15} & 0 \\
d_{15} & 0 & 0
\end{bmatrix},
\]

\[
[\delta] = \begin{bmatrix}
\delta_{11} & 0 & 0 \\
0 & \delta_{22} & 0 \\
0 & 0 & \delta_{33}
\end{bmatrix}.
\]

2.4 Further assumptions

2.4.1 Actuator equation
The actuators are voltage driven. The applied electric field is assumed to be unequal zero only in transverse direction and constant in each piezoelectric layer. Considering \(0E_i = \partial\phi/\partial\Theta^i\), the following can be written for the \(k^{th}\) piezoelectric layer:

\[
\{0E\}_k = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

where \(h_k\) is the thickness and \(\phi_k\) the applied voltage of the \(k^{th}\) piezoelectric layer.

2.4.2 Sensor equation
Generally the open circuit sensor voltage can be calculated as \(V^S = q^S/C^S\), where \(q^S_k = \int_{\Omega_k} D^3 0d\Omega_k\) is the charge developed at the electrodes and \(C^S_k = \delta_{33} 0\Omega_k/h_k\) is the permittivity of the piezoelectric layer. The electric displacement...
Table 1: Geometrical and material data of the bimorph beam.

<table>
<thead>
<tr>
<th>Dimensions [mm]</th>
<th>$E$ [GPa]</th>
<th>$\nu$ [-]</th>
<th>$d_{31}$ [Cb/N]</th>
<th>$\delta_{33}$ [F/m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[100 \times 5 \times 1]$</td>
<td>2.0</td>
<td>0.29</td>
<td>$2.2 \cdot 10^{-11}$</td>
<td>$1.062 \cdot 10^{-10}$</td>
</tr>
</tbody>
</table>

$D^3$ is assumed to be only depending on the mechanical strains and not on the self-generated electric field. According to the direct piezoelectric effect equation (7) it can therefore be calculated as

$$D^3 = d_{31} \left[ \varepsilon_{11} \left( c_{11} + c_{12} \right) + \varepsilon_{22} \left( c_{12} + c_{22} \right) + 2\varepsilon_{12} \left( c_{13} + c_{23} \right) \right]. \quad (10)$$

### 2.5 Finite element formulation

In the volume integral of the internal virtual work the integration through the thickness can be performed analytically, which reduces the 3-D problem into a 2-D one. Expression (5) can now be written as:

$$2\delta W_i = \int_{\Omega} \left[ \sum_{n=0}^{2} 2^n L^{\alpha\beta} \delta_{0}^{2} \varepsilon_{\alpha\beta} + \sum_{n=0}^{1} 2^n L^{\alpha3} \delta_{0}^{2} \varepsilon_{\alpha3} \right] 0d\Omega, \quad (11)$$

where the stress resultant components $\frac{2^n}{0} L^{\alpha\beta}$ and $\frac{2^n}{0} L^{\alpha3}$ are described in [13]. The $2^{nd}$ Piola-Kirchhoff stress resultant components in configuration $2^C$ can now be decomposed into the stress resultant components from the first right hand side terms of equation (8), $\frac{2^n}{0} L^{m\beta}$, which denote the mechanically generated stress, and the second right hand side terms, $\frac{2^n}{0} L^{\alpha3}$, which denote the electrically generated stress resultant components. Further $\frac{2^n}{0} L^{kl}$ can be decomposed into the stress terms for configuration $1^C$, $\frac{1^n}{0} L^{kl}$, and an increment of the stress terms $0 L^{kl}$. The mechanical part of the internal virtual work can be written as

$$2\delta W_i = \{\delta q\}^T \left( \left[ \frac{1}{0} K u \right] + \left[ \frac{1}{0} K g \right] \right) \cdot \{q\}, \quad (12)$$

where $\{\frac{1}{0} F\}$ is the balanced force vector, $\left[ \frac{1}{0} K u \right]$ the first part of the incremental stiffness matrix, $\left[ \frac{1}{0} K g \right]$ the geometrical stiffness matrix and $\{q\}$ the vector of incremental nodal displacements.

The electric part of the internal virtual work can be summarised as an external force vector $\{\delta q\} \{\frac{2}{0} R e\}$, where $\{\frac{2}{0} R e\}$ is calculated as

$$\{\frac{2}{0} R e\} = \int_{\Omega} \left( [B_L]^T + 2 [B_{NL}]^T \right) \{\frac{2}{0} L_e\} d\Omega. \quad (13)$$

The matrices $[B_L]$ and $[B_{NL}]$ are the strain displacement matrices for the linear and the nonlinear part, respectively, and $\{\frac{2}{0} L_e\}$ is calculated as

$$\{\frac{2}{0} L_e\} = \sum_{k=1}^{np} \left\{ \left\{ \frac{0}{0} L_e \right\}_k^T \left\{ \frac{1}{0} L_e \right\}_k^T \left\{ \frac{2}{0} L_e \right\}_k^T \right\} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \quad (14)$$
where

\[
\{ n L \}_k^T = \frac{d_{31}}{\theta_k} \int_{h_k} \left\{ \left( c_{11} + c_{12} \right) \left( c_{12} + c_{22} \right) \left( c_{13} + c_{23} \right) \right\} \left( \Theta^3 \right)^n 0 c d \Theta^3 ,
\]

\(0c\) is the shifter tensor determinant and \( np \) is the number of piezoelectric layers. Introducing the external (mechanical) force vector \( \{ 0 R \} \) the following equation needs to be solved with e.g. a Newton-Raphson method

\[
\left( \left[ \begin{array}{c} 0 K_u \\ 0 K_g \end{array} \right] + \left[ \begin{array}{c} 0 R_e \\ 0 F \end{array} \right] \right) \cdot \{ q \} = \left[ \begin{array}{c} 0 R \end{array} \right] + \left[ \begin{array}{c} 0 R_e \end{array} \right] - \left[ \begin{array}{c} 0 F \end{array} \right].
\]

(15)

3 Results and discussion

The effect of the geometrically nonlinear formulation has been tested on the actuator and sensor properties of the proposed finite element. Therefor two examples have been chosen from literature which are the bimorph beam originally proposed by Tzou and Tseng [4] and the C-block actuators of Moskalik and Brei [14].

Figure 2(a) displays the results in case of a cantilever bimorph beam, with material and geometrical properties described in table 1, deformed with a tip deflection of 1cm. Five equal pairs of sensor electrodes are attached to the upper and lower side of the beam and the resulting open circuit voltage is displayed. One notices that the linear solution agrees with the results obtained by Piefort [3]. Now comparing the moderate rotation results with these linear results it is important to decide whether the sensor is meant to be a force or displacement sensing device. Using the same force, as was applied in the linear case, will result in a smaller tip displacement. If one wants to obtain the same displacement, a larger force needs to be applied. In both cases the sensor voltage of the most left sensor is significantly different from the linear case. This can be explained by the high membrane stress
concentration near the clamping point, which makes differences in the obtained results more pronounced.

Figure 2(b) shows the displacement of the same piezoelectric bimorph beam when an electric voltage of 1V is applied across the thickness. The tip displace-
ment is in such a small range that one does not expect any difference between the linear and the nonlinear case, which is confirmed in figure 2(b). The difference between the analytical solution: \( \frac{2d_{31} \phi x^2}{3h^2} \), and the finite element analysis is explained by the high stress concentration due to the clamping condition in the finite element analysis. This phenomenon does not occur in the beam theory on which the analytical solution is based.

Figure 3 displays the voltage/deflection curves of C-block actuators described by Moskalik and Brei [14] and Piefort [3]. The comparison between the linear and the moderate rotation results show again that the geometrical nonlinearity bears no significance when piezoelectric actuators are considered. One notices that in the first two cases the present results differ slightly from the results obtained by Piefort [3]. Looking at the mentioned author’s modelling this difference is explained by the negligence of the empty spaces between the piezoelectric layers. This results in a stiffer model. The considerable difference in the third case could not be explained by the present authors.

4 Conclusions

A finite element code has been developed to model shell structures with integrated piezoelectric layers. The geometrically nonlinear strain-displacement formulation is based on the moderate rotation shell theory. A proper definition of the required material electric field values was introduced.

The code was validated by means of two commonly used benchmark problem: the piezoelectric bimorph and the C-block actuator of Moskalik and Brei [14]. Further it was shown that for actuation the effect of membrane stresses does not bear as much effect as for instance the clamping stresses do in comparison with a beam theory. Rather than the actuation properties of a smart piezointegrated structure, the sensing properties are much more affected by the application of a geometrically nonlinear formulation.

References


